On the Continuability of Solutions of Bidimensional Systems

JUAN E. NÁPOLES V.

I.S.P.H., Matemáticas, Holguín 81000, Cuba
(Presented by W. Oktasinski)

To the memory of José Antonio Repilado Ramírez

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1. INTRODUCTION

Consider the system:

\[ \begin{align*}
    x' &= \alpha(y) - \beta(y)F(x), \\
    y' &= -a(t)g(x),
\end{align*} \tag{1} \]

where \( \alpha, \beta, f, F(x) = \int_0^x f(s)ds \) are real valued continuous functions on \( \mathbb{R} \). Moreover \( a \) is continuous on \( [0, +\infty) \) and \( g: \mathbb{R} \to \mathbb{R} \) is continuous such that \( xg(x) > 0 \) for \( x > 0 \). Dots denote the differentiation with respect to \( t \). Let us note that in the case \( \alpha(y) = y, \beta(y) \equiv 1 \) system (1) reduces to the equation of the second order:

\[ x'' + f(x)x' + a(t)g(x) = 0. \tag{2} \]

If \( a \equiv 1 \) and \( f \equiv h' \) then (2) is the well known Liénard’s equation:

\[ x'' + h'(x)x' + g(x) = 0. \]

It is known that in the case that \( a \) is sufficiently smooth positive, all solutions of (2) can be extended for all \( t > 0 \) (see [5]). The case of negative \( a \) is quite different. If \( a \) is negative at some point then the equation (2) without damping, i.e.

\[ x'' + a(t)g(x) = 0, \]

has solutions which are not continuable for all \( t > 0 \). Moreover for \( a \) as above nothing is known about continuability of solutions of (2). The aim of this
short note is to study the continuability of solutions of system (1) in the case of a negative in one point.

Let us note that results presented in this note complete earlier papers ([2-4]).

2. CONTINUABILITY OF SOLUTIONS OF (2)

Let $G(x) = \int_0^x g(s) \, ds$. Now we state our first result.

**THEOREM 1.** Suppose $a(t_1) < 0$ for some $t_1 > 0$ and $F(x) \leq N$ for $x \geq 0$. If the following condition is fulfilled:

\[(3) \quad \int_0^{+\infty} (1 + G(x))^{-\frac{1}{2}} \, dx < \infty,\]

then (2) has solution $x(t)$ which is not continuable to $+\infty$.

**THEOREM 2.** Let $a(t)$ be continuous satisfying $a(t) < 0$ on an interval $t_1 \leq t \leq t_2$ with $a(t_2) \leq 0$ and there exists $N > 0$ such that $0 \leq F(x) \leq N$ for $x \geq 0$. Then (2) has a solution $(x(t), y(t))$ defined for $t = t_1$ satisfying:

\[(4) \quad \lim_{t \to T^-} |x(t)| = +\infty,\]

for some $T \in (t_1, t_2]$ if and only if (3) holds.

The proofs of Theorems 1 and 2 are based on ideas of proofs presented in [1]. In proofs of [1] it is considered the system

\[
x' = y',
\]

\[
y' = -a(t)g(x).
\]

To prove Theorems 1 and 2 we consider, instead of the above system the following one:

\[
x' = y - F(x),
\]

\[
y' = -a(t)g(x).
\]

Modifying ideas of [1] we obtain the expected results.

**Remark 1.** The case $x \leq 0$ can be proved in a similar way, using the condition $-N \leq F(x) \leq 0$ for $x \leq 0$ and

\[(5) \quad \int_0^{-\infty} (1 + G(x))^{-\frac{1}{2}} \, dx > -\infty,\]

thus a similar argument may be given in quadrant III of the Phase Plane.
3. The General System (1)

To consider system (1) we assume additionally: a) $\alpha$ is strictly increasing and such that $y_{0}\alpha(y) > 0$ if $y \neq 0$, b) $\beta(y) > 0$, c) $xF(x) > 0$ if $x \neq 0$, d) $g$ is an increasing function on $(-\infty, +\infty)$. From the Theorem 1 we obtain:

**Theorem 3.** Let $a(t_1) < 0$ for some $t_1 > 0$. Suppose there exists $\delta_1 > 0$ and $K > 0$ such that for $t_1 \leq t \leq t_1 + \delta$ either:

\[ y - F(x) \leq \alpha(y) - \beta(y)F(x) \quad \text{for } x > K \text{ and } y > K, \]

or

\[ y - F(x) \geq \alpha(y) - \beta(y)F(x) \quad \text{for } x < -K \text{ and } y < -K. \]

If (6) and (3) hold or (7) and (5) hold, then (1) has solutions $(x(t), y(t))$ satisfying $|x(t)| \to \infty$ for some $T_1 > t_1$.

**Remark 2.** The formulation of the comparison result corresponding to Theorem 2 is easy and we leave it to the reader.

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**References**


