On a Property of Gram’s Determinant

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1. Introduction

Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space over the real or complex number field \(\mathbb{K}\) and \(\{x_1, \ldots, x_n\}\) a system of vectors in \(H\). Consider the Gram matrix

\[ G(x_1, \ldots, x_n) := [(x_i, x_j)]_{i,j=1}^n \]

and the Gram determinant

\[ \Gamma(x_1, \ldots, x_n) := \det G(x_1, \ldots, x_n). \]

The following inequality is well known in the literature as Gram’s inequality (see, e.g., [4, p. 595]):

\[ \Gamma(x_1, \ldots, x_n) \geq 0. \]  

Note that equality holds in (1.1) iff the system of vectors \(\{x_1, \ldots, x_n\}\) is linearly dependent.

A well known converse of this inequality is the following:

\[ \Gamma(x_1, \ldots, x_n) \leq \prod_{i=1}^n \|x_i\|^2, \quad x_i \in H \ (i = 1, n) \]  

known as Hadamard’s inequality. Equality holds in (1.2) if and only if \((x_i, x_j) = \delta_{i,j}\|x_i\|\|x_j\|\) for all \(i, j \in \{1, \ldots, n\}\) (see [3]).

Some special but very interesting inequalities which involve Gram determinants are the following (see [4, p. 597]):

\[ \frac{\Gamma(x_1, \ldots, x_n)}{\Gamma(x_1, \ldots, x_k)} \leq \frac{\Gamma(x_2, \ldots, x_n)}{\Gamma(x_2, \ldots, x_k)} \leq \cdots \leq \Gamma(x_{k+1}, \ldots, x_n) \]  

\[ \Gamma(x_1, \ldots, x_n) \leq \Gamma(x_1, \ldots, x_k)\Gamma(x_{k+1}, \ldots, x_n) \]
(1.5) \[ \left[ \Gamma(x_1 + y_1, x_2, \ldots, x_n) \right]^{1/2} \leq \left[ \Gamma(x_1, x_2, \ldots, x_n) \right]^{1/2} + \left[ \Gamma(y_1, x_2, \ldots, x_n) \right]^{1/2}. \]

2. Results

For a fixed inner product \((\cdot, \cdot)\) on linear space \(H\) over \(\mathbb{K}\), consider the Gram determinant
\[ \Gamma((\cdot, \cdot); x_1, \ldots, x_n) = \det \left[ (x_i, x_j) \right]_{i,j=1,n} \cdot \]

We now state and prove our main result.

**Theorem 2.1.** Let \((\cdot, \cdot)_1, (\cdot, \cdot)_2\) be two inner products on the linear space \(H\). Then one has the inequality:

(2.1) \[ \left[ \Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_1, \ldots, x_n) \right]^{1/2} \geq \left[ \Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) \right]^{1/2} + \left[ \Gamma((\cdot, \cdot)_2; x_1, \ldots, x_n) \right]^{1/2} \geq 0 \]

for all \(x_i \in H\) \((i = 1, n)\) and \(n \geq 2\).

**Proof.** If \(\{x_1, \ldots, x_n\}\) is a system of linearly dependent vectors in \(H\), the inequality becomes an identity.

Suppose that \(\{x_1, \ldots, x_n\}\) is linearly independent. With this assumption we can consider the map:
\[ \gamma((\cdot, \cdot); x_1, \ldots, x_n) := \frac{\Gamma((\cdot, \cdot); x_1, \ldots, x_n)}{\Gamma((\cdot, \cdot); x_2, \ldots, x_n)}, \quad n \geq 2, \]

where \((\cdot, \cdot)\) is an inner product on \(H\).

It is well known that (see also [3] or [4]) we have the representation
\[ \gamma((\cdot, \cdot); x_1, \ldots, x_n) = d^2(x_1, H_{x,n}) = \inf_{x \in H_{x,n}} \|x_1 - x\|^2 \]

where \(H_{x,n}\) is the linear space spanned by the linearly independent system of vectors \(\{x_2, \ldots, x_n\}\).

Let us prove the inequality:

(2.2) \[ \gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_1, \ldots, x_n) \geq \gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) + \gamma((\cdot, \cdot)_2; x_1, \ldots, x_n), \]
where \((\cdot, \cdot)_i\) is as above, which is interesting in itself (see also [1]). We have:

\[
\gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_1, \ldots, x_n) = \inf_{x \in H_{\mathbb{R}^n}} \left[ \|x_1 - x\|_1^2 + \|x_2 - x\|_2^2 \right]
\geq \inf_{x \in H_{\mathbb{R}^n}} \|x_1 - x\|_1^2 + \inf_{x \in H_{\mathbb{R}^n}} \|x_2 - x\|_2^2
= \gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) + \gamma((\cdot, \cdot)_2; x_1, \ldots, x_n),
\]

i.e., the inequality (2.2).

We now prove another inequality which is also interesting in itself [1]:

\[
(2.3) \quad \Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_1, \ldots, x_n)
\geq \Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) + \Gamma((\cdot, \cdot)_2; x_1, \ldots, x_n)
\]

for all \((\cdot, \cdot)_i\) \((i = 1, 2)\) two inner products on \(H\) and \(\{x_1, \ldots, x_n\} \subset H, n \geq 2\).

Of course, we must only prove the result in the case when \(\{x_1, \ldots, x_n\}\) is linearly independent. We give a proof based on mathematical induction.

Let \(n = 2\). Then we have:

\[
\Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_1, x_2) = (\|x_1\|_1^2 + \|x_2\|_2^2)(\|x_2\|_1^2 + \|x_2\|_2^2)
- \|(x_1, x_2)\|_1^2 + \|(x_1, x_2)\|_2^2
\geq \|x_1\|_1^2\|x_2\|_2^2 + \|x_1\|_2^2\|x_2\|_1^2 + \|x_1\|_1^2\|x_2\|_2^2 + \|x_1\|_2^2\|x_2\|_1^2
- \|(x_1, x_2)\|_1^2 - 2|(x_1, x_2)_1(x_1, x_2)_2| - |(x_1, x_2)_1|^2
= \Gamma((\cdot, \cdot)_1; x_1, x_2) + \Gamma((\cdot, \cdot)_2; x_1, x_2) + \|x_1\|_1^2\|x_2\|_2^2
+ \|x_2\|_1^2\|x_2\|_2^2 - 2|(x_1, x_2)_1(x_1, x_2)_2|
\geq \Gamma((\cdot, \cdot)_1; x_1, x_2) + \Gamma((\cdot, \cdot)_2; x_1, x_2)
+ (\|x_1\|_1\|x_2\|_2 - \|x_1\|_2\|x_2\|_1)^2
\geq \Gamma((\cdot, \cdot)_1; x_1, x_2) + \Gamma((\cdot, \cdot)_2; x_1, x_2)
\]

since, by Schwarz's inequality,

\[
\|x_1\|_1\|x_2\|_2 \geq \|(x_1, x_2)\|_1, \quad \|x_1\|_2\|x_2\|_2 \geq \|(x_1, x_2)\|_2
\]

for all \(x_1, x_2 \in H\) and \((\cdot, \cdot)_i\) \((i = 1, 2)\) as above.

Now, suppose that the inequality (2.3) is true for all \((n - 1)\) linearly independent vectors in \(H\). If \(\{x_1, \ldots, x_n\} \subset H\) is linearly independent, then by
the inequality (2.2) we have:

\[
\Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_1, \ldots, x_n) \\
= \gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_1, \ldots, x_n) \Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_2, \ldots, x_n) \\
\geq \gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) \Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_2, \ldots, x_n) \\
+ \gamma((\cdot, \cdot)_2; x_1, \ldots, x_n) \Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_2, \ldots, x_n) \\
= \Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) \Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_2, \ldots, x_n) \\
\Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)
\]

(2.4)

But, by the inductive hypothesis we have

\[
\Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_2, \ldots, x_n) \geq \Gamma((\cdot, \cdot)_1; x_2, \ldots, x_n) + \Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)
\]

which shows us that

\[
\frac{\Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_2, \ldots, x_n)}{\Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)} \geq 1, \quad i = 1, 2
\]

and thus, by the inequality (2.4), we obtain the superadditivity of \(\Gamma(\cdot; x_1, \ldots, x_n)\), i.e., the inequality (2.3).

Now, by the inequality (2.4) and by this superadditivity we can prove more, i.e.

\[
\Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_1, \ldots, x_n) \\
\geq \Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) \frac{\Gamma((\cdot, \cdot)_1; x_2, \ldots, x_n) + \Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)}{\Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)} \\
+ \Gamma((\cdot, \cdot)_2; x_1, \ldots, x_n) \frac{\Gamma((\cdot, \cdot)_1; x_2, \ldots, x_n) + \Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)}{\Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)} \\
= \Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) + \Gamma((\cdot, \cdot)_2; x_1, \ldots, x_n) \\
+ \Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) \frac{\Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)}{\Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)} \\
+ \Gamma((\cdot, \cdot)_2; x_1, \ldots, x_n) \frac{\Gamma((\cdot, \cdot)_1; x_2, \ldots, x_n)}{\Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)}.
\]
But a simple inequality for real numbers shows us:
\[
\Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) \frac{\Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)}{\Gamma((\cdot, \cdot)_1; x_2, \ldots, x_n)} \\
+ \Gamma((\cdot, \cdot)_2; x_1, \ldots, x_n) \frac{\Gamma((\cdot, \cdot)_1; x_2, \ldots, x_n)}{\Gamma((\cdot, \cdot)_2; x_2, \ldots, x_n)} \\
\geq 2\Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) \Gamma((\cdot, \cdot)_2; x_1, \ldots, x_n)^{1/2}
\]
which gives, from inequality (2.5), that
\[
\Gamma((\cdot, \cdot)_1 + (\cdot, \cdot)_2; x_1, \ldots, x_n) \\
\geq (\Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n)^{1/2} + \Gamma((\cdot, \cdot)_2; x_1, \ldots, x_n)^{1/2})^2
\]
i.e., the desired inequality (2.1). □

**Remark 2.2.** If the system \(\{x_1, \ldots, x_n\}\) is linearly independent and \((\cdot, \cdot)_2 > (\cdot, \cdot)_1\), i.e. \(\|x\|_2 > \|x_1\|\) for all \(x \in H \setminus \{0\}\), we have the monotonicity property [1]:
\[
(2.6) \quad \gamma((\cdot, \cdot)_2; x_1, \ldots, x_n) \geq \gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) \geq 0.
\]
Indeed, by the inequality (2.2), we have:
\[
\gamma((\cdot, \cdot)_2; x_1, \ldots, x_n) = \gamma((\cdot, \cdot)_{2,1} + (\cdot, \cdot)_1; x_1, \ldots, x_n) \\
\geq \gamma((\cdot, \cdot)_{2,1}; x_1, \ldots, x_n) + \gamma((\cdot, \cdot)_1; x_1, \ldots, x_n)
\]
where \((\cdot, \cdot)_{2,1} = (\cdot, \cdot)_2 - (\cdot, \cdot)_1\) is an inner product on \(H\).

**Remark 2.3.** If \((\cdot, \cdot)_2 \geq (\cdot, \cdot)_1\), i.e. \(\|x\|_2 \geq \|x_1\|\) for all \(x \in H\), and \(\{x_1, \ldots, x_n\} \subset H\), then
\[
(2.7) \quad \Gamma((\cdot, \cdot)_2; x_1, \ldots, x_n) \geq \Gamma((\cdot, \cdot)_1; x_1, \ldots, x_n) \geq 0
\]
i.e. the monotonicity of Gram’s determinant [1].

For other related results, see the papers [1] and [2] where further applications and consequences are given.

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REFERENCES