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Tauberian Operators on Spaces of Integrable Functions†

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We study tauberian operators from $L_1(\mu)$ into a Banach space Y, where μ is a non-purely atomic, finite measure. In the case in which μ is a purely atomic measure, our results are also valid but trivial, because in this case $L_1(\mu)$ has the Schur property: weakly convergent sequences are convergent.

In Section 1, we characterize the tauberian operators $T : L_1(\mu) \longrightarrow Y$ as those operators T such that $\liminf_n \|Tf_n\| > 0$ for every disjoint normalized sequence (f_n) in $L_1(\mu)$; or equivalently, the kernel $N(T^{**})$ of the second conjugate of T coincides with $N(T)$. We show that the class of all tauberian operators from $L_1(\mu)$ into Y is open in the class of all operators, and we give several examples of tauberian operators $T : L_1(\mu) \longrightarrow Y$. As a consequence, we prove that $L_1(\mu)$ is contained isomorphically in every quotient of $L_1(\mu)$ by any of its reflexive subspaces.

In Section 2, we show that if an operator $T : L_1(\mu) \longrightarrow Y$ is tauberian, then it is supertauberian (the ultrapowers T_u of T are tauberian operators) and its second conjugate T^{**} is also tauberian. Moreover we characterize T tauberian in terms of the kernel $N(T_u)$ of any of its non-trivial ultrapowers.

Section 3 is devoted to the action of tauberian operators on the dyadic tree of $L_1[0,1]$. We prove that an operator $T : L_1[0,1] \longrightarrow Y$ is tauberian if and only if for every sequence (f_n) contained in the dyadic tree on $L_1[0,1]$ and equivalent to the unit vector basis of ℓ_1, the sequence (Tf_n) is also equivalent to the unit vector basis of ℓ_1 up to a finite quantity of elements.

In Section 4 we study the admissible perturbations for tauberian operators on $L_1(\mu)$. It is known that the class of tauberian operators is stable under perturbation by weakly compact operators. We identify the perturbation class

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with the class of weakly precompact operators, by showing that an operator $K : L_1(\mu) \longrightarrow Y$ is weakly precompact if and only if $T + K$ is tauberian for every tauberian operator $T : L_1(\mu) \longrightarrow Y$.

We use standard notations: X and Y are Banach spaces, B_X the closed unit ball of X, S_X the unit sphere of X, $\mathcal{B}(X,Y)$ the class of bounded linear operators from X into Y, X^* the dual of X, $T^* : Y^* \longrightarrow X^*$ the conjugate operator of $T \in \mathcal{B}(X,Y)$, and $R(T)$ and $N(T)$ the range and kernel of T. We identify X with a subspace of X^{**}. We denote the positive integers by $\mathbb{N}$, and the real numbers by $\mathbb{R}$.

1. Disjoint sequences in $L_1(\mu)$

Let (Ω, Σ, μ) be a non-purely atomic finite measure space. We call $A \in \Sigma$ *non-purely atomic* if A is not a union of atoms.

DEFINITION 1. ([7]) An operator $T \in \mathcal{B}(X,Y)$ is *tauberian* if $T^{**^{-1}}(Y) \subset X$; equivalently [7, Th. 3.2], if any bounded sequence $(x_n) \subset X$ admits a weakly convergent subsequence (x_{n_k}) whenever (Tx_n) is weakly convergent.

We denote the class of all tauberian operators from X into Y by $\mathcal{T}(X,Y)$. Next we give the main result of this section.

THEOREM 2. *For $T \in \mathcal{B}(L_1(\mu), Y)$, the following statements are equivalent:*

(1) *T is tauberian;*
(2) $N(T) = N(T^{**})$;
(3) $\liminf_n \|Tf_n\| > 0$ *for every normalized disjoint sequence (f_n) in $L_1(\mu)$;*
(4) *there exists $r > 0$ such that $\liminf_n \|Tf_n\| > r$ for every normalized disjoint sequence (f_n) in $L_1(\mu)$.*

COROLLARY 3. *The class $\mathcal{T}(L_1(\mu), Y)$ is open in $\mathcal{B}(L_1(\mu), Y)$.*

Note that the class $\mathcal{T}(X,Y)$ is not open in general [1], [12]. Now we present some examples of tauberian operators on $L_1(\mu)$.

EXAMPLES. *a)* If R is a reflexive subspace of $L_1(\mu)$, the quotient operator $Q : L_1(\mu) \longrightarrow L_1(\mu)/R$ is tauberian. The space $L_1[0,1]$ contains a large list of reflexive subspaces. For instance, the closed space generated by the

Rademacher functions is isomorphic to ℓ_2 [9, Th. 2.b.3]. Also, it is known [10, Th. 2.f.5] that for $1 < r < 2$ there exists a closed subspace $M_r \subset L_1[0,1]$ isomorphic to $L_r[0,1]$. Note that none of those reflexive subspaces M_r contains the remaining M_p for $1 < p < 2$, because the type is hereditary by subspaces.

b) If $K \in \mathcal{B}(X,Y)$ is weakly compact and $T \in \mathcal{B}(X,Y)$ is tauberian, it is easy to check that $T + K$ is also tauberian.

c) Given a reflexive subspace $R \subset L_1(\mu)$, the distance from the quotient operator $Q : L_1(\mu) \longrightarrow L_1(\mu)/R$ to the boundary of $\mathcal{T}(L_1(\mu), L_1(\mu)/R)$ is equal to 1. Thus, if $S \in \mathcal{B}(L_1(\mu), L_1(\mu)/R)$ and $\|S\| < 1$ then $Q + S$ is tauberian.

Given a measurable subset $C \subset \Omega$ with $\mu(C) > 0$, we denote by $L_1(C)$ the subspace of $L_1(\mu)$ which consists of all functions f such that $f = \chi_C f$. If C is non-purely atomic then $L_1(C)$ is isomorphic to $L_1(\mu)$.

PROPOSITION 4. *Let $T \in \mathcal{B}(L_1(\mu), Y)$ be a tauberian operator. For every non-purely atomic measurable set $A \subset \Omega$ with $\mu(A) > 0$ there is a non-purely atomic subset $C \subset A$ with $\mu(C) > 0$ so that the restriction $T|_{L_1(C)}$ is an isomorphism.*

COROLLARY 5. *The class $\mathcal{T}(L_1(\mu), Y)$ is non-empty if and only if Y contains a subspace isomorphic to $L_1(\mu)$. In particular, if M is a reflexive subspace of $L_1(\mu)$ then $L_1(\mu)/M$ contains a subspace isomorphic to $L_1(\mu)$.*

2. SUPERTAUBERIAN OPERATORS

Tacon [12] introduced the class of supertauberian operators as a refinement of the class of tauberian operators. An operator $T \in \mathcal{B}(X,Y)$ is said to be *supertauberian* if for every $0 < \varepsilon < 1$ there exists a positive integer $n \in \mathbb{N}$ for which there are not families $\{x_1, \dots, x_n\} \subset S_X$, $\{f_1, \dots, f_n\} \subset S_{X^*}$ verifying $f_k(x_m) > \varepsilon$ for $1 \le k \le m \le n$, $f_k(x_m) = 0$ for $1 \le m < k \le n$ and $\|Tx_k\| < 1/k$ for $k = 1, \dots, n$.

Supertauberian operators can be characterized in terms of ultrapowers of Banach spaces [4]. In order to be precise we need to introduce some notation.

An ultrafilter $\mathcal{U}$ on an infinite set I is said to be *countably incomplete* if there is a countable partition $\{I_n : n \in \mathbb{N}\}$ of I verifying $I_n \notin \mathcal{U}$ for all $n \in \mathbb{N}$. Henceforth, $\mathcal{U}$ will be a fixed countably incomplete ultrafilter on an infinite set I.

Consider the Banach space $\ell_\infty(I,X)$ which consists of all bounded families $(x_i)_{i\in I}$ in X endowed with the norm $\|(x_i)\|_\infty := \sup\{\|x_i\| : i \in I\}$. Let $N_\mathcal{U}(X)$ be the closed subspace of all families $(x_i) \in \ell_\infty(I,X)$ which converge to 0 following $\mathcal{U}$. The *ultrapower of* X *following* $\mathcal{U}$ is defined as the quotient

$$X_\mathcal{U} := \frac{\ell_\infty(I,X)}{N_\mathcal{U}(X)}.$$

The element of $X_\mathcal{U}$ including the family $(x_i) \in \ell_\infty(I,X)$ as a representative is denoted by $[x_i]$. The ultrapower $X_\mathcal{U}$ contains canonically an isometric copy of X generated by the constant families of $\ell_\infty(I,X)$. We identify this copy with X. An operator $T \in \mathcal{B}(X,Y)$ has an extension $T_\mathcal{U} \in \mathcal{B}(X_\mathcal{U}, Y_\mathcal{U})$ given by $T_\mathcal{U}[x_i] = [Tx_i]$.

PROPOSITION 6. *[4, Th. 9] An operator $T \in \mathcal{B}(X,Y)$ is supertauberian if and only if $T_\mathcal{U}$ is tauberian.*

The ultrapower $L_1(\mu)_\mathcal{U}$ was studied extensively by Heinrich [5]. For the convenience of the reader, we give a description of it.

Let $\mathcal{B}(I,\Omega)$ be the set of all families $(x_i)_{i\in I}$ such that $x_i \in \Omega$. The ultrafilter $\mathcal{U}$ induces an equivalence relation $\sim$ on $\mathcal{B}(I,\Omega)$ given by $(x_i)_{i\in I} \sim (y_i)_{i\in I}$ if $\{i \in I : x_i = y_i\} \in \mathcal{U}$. We write

$$\Omega^\mathcal{U} := \frac{\mathcal{B}(I,\Omega)}{\sim},$$

and $(x_i)^\mathcal{U}$ denotes the element of $\Omega^\mathcal{U}$ whose representative is $(x_i)_{i\in I}$. If $\{A_i : i \in I\}$ is a family of subsets of Ω, we write $(A_i)^\mathcal{U} := \{(x_i)^\mathcal{U} : x_i \in A_i\}$. The Boolean algebra $\Sigma_\mathcal{U}$ on $\Omega^\mathcal{U}$, defined by $\Sigma_\mathcal{U} := \{(A_i)^\mathcal{U} : A_i \in \Sigma\}$, generates a σ-algebra $\Gamma_\mathcal{U}$. The measure μ induces a measure $\mu_\mathcal{U}$ on $\Gamma_\mathcal{U}$, univocally defined by its value on the elements $(A_i)^\mathcal{U} \in \Sigma_\mathcal{U}$, given by $\mu_\mathcal{U}((A_i)^\mathcal{U}) := \lim_\mathcal{U} \mu(A_i)$.

There exists a measure space (Θ, Υ, ν) such that $L_1(\mu)_\mathcal{U} \cong L_1(\mu_\mathcal{U}) \oplus_1 L_1(\nu)$.

Recall that a Banach space X is *superreflexive* if every Banach space finitely representable in X is reflexive; equivalently, if any ultrapower $X_\mathcal{U}$ is reflexive [5].

From a result of Rosenthal [11], it follows that all reflexive subspaces of $L_1(\mu)$ are superreflexive. Here we obtain this fact from the following result.

PROPOSITION 7. *Let E be a subspace of $L_1(\mu)$. Then E is reflexive if and only if $E_\mathcal{U}$ is contained in $L_1(\mu_\mathcal{U})$.*

COROLLARY 8. *An operator $T \in \mathcal{B}(L_1(\mu), Y)$ is tauberian if and only if T is supertauberian. In this case, $T^{(2n)}$ is tauberian for all $n \in \mathbb{N}$.*

For $T \in \mathcal{B}(L_1(\mu), Y)$, we give a better result.

PROPOSITION 9. *An operator $T \in \mathcal{B}(L_1(\mu), Y)$ is tauberian if and only if $N(T_u) \subset L_1(\mu_u)$.*

3. Tauberian operators and the dyadic tree

In this section we characterize tauberian operators $T : L_1[0,1] \longrightarrow Y$ by its action over the dyadic tree of $L_1[0,1]$. A *tree* in a Banach space Y is a bounded family

$$\{y_k^n : n = 0, 1, 2, \dots ; 1 \le k \le 2^n\} \subset Y$$

such that $y_k^n = 2^{-1}(y_{2k-1}^{n+1} + y_{2k}^{n+1})$ for all n and k. An example is the so called *dyadic tree* on $L_1[0,1]$, given by

$$\chi_k^n := 2^n \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}.$$

The intervals $(\frac{k-1}{2^n}, \frac{k}{2^n})$ are called *dyadic.* Any operator $T \in \mathcal{B}(L_1[0,1], Y)$ determines a tree on Y given by $y_k^n := T\chi_k^n$. Conversely, a tree $(y_k^n) \subset Y$ determines an operator in $\mathcal{B}(L_1[0,1], Y)$. We refer to [2] for the details.

THEOREM 10. *An operator $T \in \mathcal{B}(L_1[0,1], Y)$ is tauberian if and only if for every sequence (x_n) contained in the dyadic tree of $L_1[0,1]$ and equivalent to the unit vector basis of ℓ_1, there is a $n_0 \in \mathbb{N}$ such that $(Tx_n)_{n \ge n_0}$ is equivalent to the unit vector basis of ℓ_1.*

4. The perturbation class of $\mathcal{T}(L_1(\mu), Y)$

For a Banach space $\mathcal{A}$ and a subset $\mathcal{S} \subset \mathcal{A}$, Lebow and Schechter [8] define *the perturbation class of $\mathcal{S}$ in $\mathcal{A}$* as the set

$$P(\mathcal{S}) := \{a \in \mathcal{A} : a + s \in \mathcal{S} \text{ for all } s \in \mathcal{S}\}.$$

We say that $\mathcal{C} \subset \mathcal{A}$ is an *admissible class for $\mathcal{S}$ if $\mathcal{C} \subset P(\mathcal{S})$*. Here we study the perturbation class of $\mathcal{T}(L_1(\mu), Y)$ in $\mathcal{B}(L_1(\mu), Y)$.

For $\mathcal{S} = \mathcal{T}(X, Y)$, the class $WCo(X, Y)$ of all weakly compact operators from X into Y is an admissible class [12]. Moreover, a broader admissible class for $\mathcal{T}(X,Y)$ can be introduced as follows. An operator $T \in \mathcal{B}(X,Y)$ is said to be *R-strictly singular* if for any operator L into X such that TL is tauberian, L is weakly compact [3]. The perturbation class $P(\mathcal{T}(X,Y))$ is not well known in general. However, for $X = L_1(\mu)$, we find that $P(\mathcal{T}(L_1(\mu),Y))$ coincides with the class $Ro(L_1(\mu),Y)$ of all weakly precompact operators. Recall that $T \in \mathcal{B}(X,Y)$ is said to be a *weakly precompact operator* if for every bounded sequence $(x_n) \subset X$, (Tx_n) contains a weakly Cauchy subsequence.

PROPOSITION 11. *Let Y be a Banach space such that $\mathcal{T}(L_1(\mu),Y) \neq \emptyset$. An operator $K \in \mathcal{B}(L_1(\mu),Y)$ is weakly precompact if and only if for every $T \in \mathcal{T}(L_1(\mu),Y)$, the operator $T+K$ is tauberian.*

Herman [6] call an operator $T \in \mathcal{B}(X,Y)$ *almost weakly compact* if given a closed subspace $H \subset X$ such that $T|_H$ is an isomorphism, one has that H is reflexive.

PROPOSITION 12. *For $T \in \mathcal{B}(L_1(\mu),Y)$, the following statements are equivalent:*

(1) *T is weakly precompact;*

(2) *T is R-strictly singular;*

(3) *T is almost weakly compact.*

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