Tauberian Operators on Spaces of Integrable Functions†

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We study tauberian operators from $L_1(\mu)$ into a Banach space $Y$, where $\mu$ is a non-purely atomic, finite measure. In the case in which $\mu$ is a purely atomic measure, our results are also valid but trivial, because in this case $L_1(\mu)$ has the Schur property: weakly convergent sequences are convergent.

In Section 1, we characterize the tauberian operators $T : L_1(\mu) \rightarrow Y$ as those operators $T$ such that $\lim \inf_n \|Tf_n\| > 0$ for every disjoint normalized sequence $(f_n)$ in $L_1(\mu)$; or equivalently, the kernel $N(T^{**})$ of the second conjugate of $T$ coincides with $N(T)$. We show that the class of all tauberian operators from $L_1(\mu)$ into $Y$ is open in the class of all operators, and we give several examples of tauberian operators $T : L_1(\mu) \rightarrow Y$. As a consequence, we prove that $L_1(\mu)$ is contained isomorphically in every quotient of $L_1(\mu)$ by any of its reflexive subspaces.

In Section 2, we show that if an operator $T : L_1(\mu) \rightarrow Y$ is tauberian, then it is supertaubernian (the ultrapowers $T_\mu$ of $T$ are tauberian operators) and its second conjugate $T^{**}$ is also tauberian. Moreover we characterize $T$ tauberian in terms of the kernel $N(T_\mu)$ of any of its non-trivial ultrapowers.

Section 3 is devoted to the action of tauberian operators on the dyadic tree of $L_1[0,1]$. We prove that an operator $T : L_1[0,1] \rightarrow Y$ is tauberian if and only if for every sequence $(f_n)$ contained in the dyadic tree on $L_1[0,1]$ and equivalent to the unit vector basis of $\ell_1$, the sequence $(Tf_n)$ is also equivalent to the unit vector basis of $\ell_1$ up to a finite quantity of elements.

In Section 4 we study the admissible perturbations for tauberian operators on $L_1(\mu)$. It is known that the class of tauberian operators is stable under perturbation by weakly compact operators. We identify the perturbation class

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with the class of weakly precompact operators, by showing that an operator $K : L_1(\mu) \to Y$ is weakly precompact if and only if $T + K$ is tauberian for every tauberian operator $T : L_1(\mu) \to Y$.

We use standard notations: $X$ and $Y$ are Banach spaces, $B_X$ the closed unit ball of $X$, $S_X$ the unit sphere of $X$, $\mathcal{B}(X,Y)$ the class of bounded linear operators from $X$ into $Y$, $X^*$ the dual of $X$, $T^* : Y^* \to X^*$ the conjugate operator of $T \in \mathcal{B}(X,Y)$, and $R(T)$ and $N(T)$ the range and kernel of $T$. We identify $X$ with a subspace of $X^{**}$. We denote the positive integers by $\mathbb{N}$, and the real numbers by $\mathbb{R}$.

1. Disjoint Sequences in $L_1(\mu)$

Let $(\Omega, \Sigma, \mu)$ be a non-purely atomic finite measure space. We call $A \in \Sigma$ non-purely atomic if $A$ is not a union of atoms.

**Definition 1.** ([7]) An operator $T \in \mathcal{B}(X,Y)$ is tauberian if $T^{**}^{-1}(Y) \subset X$; equivalently [7, Th. 3.2], if any bounded sequence $(x_n) \subset X$ admits a weakly convergent subsequence $(x_{n_k})$ whenever $(Tx_n)$ is weakly convergent.

We denote the class of all tauberian operators from $X$ into $Y$ by $\mathcal{T}(X,Y)$. Next we give the main result of this section.

**Theorem 2.** For $T \in \mathcal{B}(L_1(\mu), Y)$, the following statements are equivalent:

1. $T$ is tauberian;
2. $N(T) = N(T^{**})$;
3. $\lim \inf_n \|Tf_n\| > 0$ for every normalized disjoint sequence $(f_n)$ in $L_1(\mu)$;
4. there exists $r > 0$ such that $\lim \inf_n \|Tf_n\| > r$ for every normalized disjoint sequence $(f_n)$ in $L_1(\mu)$.

**Corollary 3.** The class $\mathcal{T}(L_1(\mu), Y)$ is open in $\mathcal{B}(L_1(\mu), Y)$.

Note that the class $\mathcal{T}(X,Y)$ is not open in general [1], [12]. Now we present some examples of tauberian operators on $L_1(\mu)$.

**Examples.** a) If $R$ is a reflexive subspace of $L_1(\mu)$, the quotient operator $Q : L_1(\mu) \to L_1(\mu)/R$ is tauberian. The space $L_1[0,1]$ contains a large list of reflexive subspaces. For instance, the closed space generated by the
Rademacher functions is isomorphic to $\ell_2$ [9, Th. 2.6.3]. Also, it is known [10, Th. 2.2.5] that for $1 < r < 2$ there exists a closed subspace $M_r \subset L_1[0,1]$ isomorphic to $L_r[0,1]$. Note that none of those reflexive subspaces $M_r$ contains the remaining $M_p$ for $1 < p < 2$, because the type is hereditary by subspaces.

b) If $K \in B(X,Y)$ is weakly compact and $T \in B(X,Y)$ is tauberian, it is easy to check that $T + K$ is also tauberian.

c) Given a reflexive subspace $R \subset L_1(\mu)$, the distance from the quotient operator $Q : L_1(\mu) \to L_1(\mu)/R$ to the boundary of $T(L_1(\mu), L_1(\mu)/R)$ is equal to 1. Thus, if $S \in B(L_1(\mu), L_1(\mu)/R)$ and $\|S\| < 1$ then $Q + S$ is tauberian.

Given a measurable subset $C \subset \Omega$ with $\mu(C) > 0$, we denote by $L_1(C)$ the subspace of $L_1(\mu)$ which consists of all functions $f$ such that $f = \chi_C f$. If $C$ is non-purely atomic then $L_1(C)$ is isomorphic to $L_1(\mu)$.

**Proposition 4.** Let $T \in B(L_1(\mu), Y)$ be a tauberian operator. For every non-purely atomic measurable set $A \subset \Omega$ with $\mu(A) > 0$ there is a non-purely atomic subset $C \subset A$ with $\mu(C) > 0$ so that the restriction $T \big|_{L_1(C)}$ is an isomorphism.

**Corollary 5.** The class $T(L_1(\mu), Y)$ is non-empty if and only if $Y$ contains a subspace isomorphic to $L_1(\mu)$. In particular, if $M$ is a reflexive subspace of $L_1(\mu)$ then $L_1(\mu)/M$ contains a subspace isomorphic to $L_1(\mu)$.

## 2. Supertauberian operators

Tacon [12] introduced the class of supertauberian operators as a refinement of the class of tauberian operators. An operator $T \in B(X,Y)$ is said to be supertauberian if for every $0 < \varepsilon < 1$ there exists a positive integer $n \in \mathbb{N}$ for which there are not families $\{x_1, \ldots, x_n\} \subset S_X$, $\{f_1, \ldots, f_n\} \subset S_X$ verifying $f_k(x_m) > \varepsilon$ for $1 \leq k \leq m \leq n$, $f_k(x_m) = 0$ for $1 \leq m < k \leq n$ and $\|Tx_k\| < 1/k$ for $k = 1, \ldots, n$.

Supertauberian operators can be characterized in terms of ultrapowers of Banach spaces [4]. In order to be precise we need to introduce some notation.

An ultrafilter $\mathcal{U}$ on an infinite set $I$ is said to be *countably incomplete* if there is a countable partition $\{I_n : n \in \mathbb{N}\}$ of $I$ verifying $I_n \notin \mathcal{U}$ for all $n \in \mathbb{N}$. Henceforth, $\mathcal{U}$ will be a fixed countably incomplete ultrafilter on an infinite set $I$. 
Consider the Banach space $\ell_\infty(I, X)$ which consists of all bounded families $(x_i)_{i \in I}$ in $X$ endowed with the norm $\|x_i\|_\infty := \sup \{\|x_i\| : i \in I\}$. Let $N_\mu(X)$ be the closed subspace of all families $(x_i) \in \ell_\infty(I, X)$ which converge to 0 following $\mathcal{U}$. The ultrapower of $X$ following $\mathcal{U}$ is defined as the quotient

$$X_\mu := \frac{\ell_\infty(I, X)}{N_\mu(X)}.$$ 

The element of $X_\mu$ including the family $(x_i) \in \ell_\infty(I, X)$ as a representative is denoted by $[x_i]$. The ultrapower $X_\mu$ contains canonically an isometric copy of $X$ generated by the constant families of $\ell_\infty(I, X)$. We identify this copy with $X$. An operator $T \in B(X, Y)$ has an extension $T_\mu \in B(X_\mu, Y_\mu)$ given by $T_\mu[x_i] = [Tx_i]$.

PROPOSITION 6. [4, Th. 9] An operator $T \in B(X, Y)$ is supertauherian if and only if $T_\mu$ is tauherian.

The ultrapower $L_1(\mu)_\mu$ was studied extensively by Heinrich [5]. For the convenience of the reader, we give a description of it.

Let $B(I, \Omega)$ be the set of all families $(x_i)_{i \in I}$ such that $x_i \in \Omega$. The ultrafilter $\mathcal{U}$ induces an equivalence relation $\sim$ on $B(I, \Omega)$ given by $(x_i)_{i \in I} \sim (y_i)_{i \in I}$ if $\{i \in I : x_i = y_i\} \in \mathcal{U}$. We write

$$\Omega^\mu := \frac{B(I, \Omega)}{\sim},$$

and $(x_i)^\mu$ denotes the element of $\Omega^\mu$ whose representative is $(x_i)_{i \in I}$. If $\{A_i : i \in I\}$ is a family of subsets of $\Omega$, we write $(A_i)^\mu := \{(x_i)^\mu : x_i \in A_i\}$. The Boolean algebra $\Sigma_\mu$ on $\Omega^\mu$, defined by $\Sigma_\mu := \{(A_i)^\mu : A_i \in \Sigma\}$, generates a $\sigma$-algebra $\Gamma_\mu$. The measure $\mu$ induces a measure $\mu_\mu$ on $\Gamma_\mu$, univocally defined by its value on the elements $(A_i)^\mu \in \Sigma_\mu$, given by $\mu_\mu((A_i)^\mu) := \lim_\mu \mu(A_i)$.

There exists a measure space $(\Theta, \mathcal{T}, \nu)$ such that $L_1(\mu)_\mu \cong L_1(\mu_\mu) \oplus_1 L_1(\nu)$.

Recall that a Banach space $X$ is superreflexive if every Banach space finitely representable in $X$ is reflexive; equivalently, if any ultrapower $X_\mu$ is reflexive [5].

From a result of Rosenthal [11], it follows that all reflexive subspaces of $L_1(\mu)$ are superreflexive. Here we obtain this fact from the following result.

PROPOSITION 7. Let $E$ be a subspace of $L_1(\mu)$. Then $E$ is reflexive if and only if $E_\mu$ is contained in $L_1(\mu_\mu)$.
COROLLARY 8. An operator \( T \in \mathcal{B}(L_1(\mu), Y) \) is tauberian if and only if \( T \) is supertauberian. In this case, \( T^{(2^n)} \) is tauberian for all \( n \in \mathbb{N} \).

For \( T \in \mathcal{B}(L_1(\mu), Y) \), we give a better result.

PROPOSITION 9. An operator \( T \in \mathcal{B}(L_1(\mu), Y) \) is tauberian if and only if \( N(T_a) \subset L_1(\mu_a) \).

3. TAUBERIAN OPERATORS AND THE DYADIC TREE

In this section we characterize tauberian operators \( T : L_1[0,1] \rightarrow Y \) by its action over the dyadic tree of \( L_1[0,1] \). A tree in a Banach space \( Y \) is a bounded family

\[ \{ y^n_k : n = 0, 1, 2, \ldots ; 1 \leq k \leq 2^n \} \subset Y \]

such that \( y^n_k = 2^{-1}(y^{n+1}_{2k-1} + y^{n+1}_{2k}) \) for all \( n \) and \( k \). An example is the so called dyadic tree on \( L_1[0,1] \), given by

\[ \chi^n_k := 2^n \chi(\frac{k}{2^n}, \frac{k}{2^n}). \]

The intervals \((\frac{k-1}{2^n}, \frac{k}{2^n})\) are called dyadic. Any operator \( T \in \mathcal{B}(L_1[0,1], Y) \) determines a tree on \( Y \) given by \( y^n_k := T \chi^n_k \). Conversely, a tree \( (y^n_k) \subset Y \) determines an operator in \( \mathcal{B}(L_1[0,1], Y) \). We refer to [2] for the details.

THEOREM 10. An operator \( T \in \mathcal{B}(L_1[0,1], Y) \) is tauberian if and only if for every sequence \( (x_n) \) contained in the dyadic tree of \( L_1[0,1] \) and equivalent to the unit vector basis of \( \ell_1 \), there is a \( n_0 \in \mathbb{N} \) such that \( (Tx_n)_{n \geq n_0} \) is equivalent to the unit vector basis of \( \ell_1 \).

4. THE PERTURBATION CLASS OF \( \mathcal{T}(L_1(\mu), Y) \)

For a Banach space \( \mathcal{A} \) and a subset \( S \subset \mathcal{A} \), Lebow and Schechter [8] define the perturbation class of \( S \) in \( \mathcal{A} \) as the set

\[ P(S) := \{ a \in \mathcal{A} : a + s \in S \text{ for all } s \in S \}. \]

We say that \( C \subset \mathcal{A} \) is an admissible class for \( S \) if \( C \subset P(S) \). Here we study the perturbation class of \( \mathcal{T}(L_1(\mu), Y) \) in \( \mathcal{B}(L_1(\mu), Y) \).

For \( S = \mathcal{T}(X,Y) \), the class \( WC_0(X,Y) \) of all weakly compact operators from \( X \) into \( Y \) is an admissible class [12]. Moreover, a broader admissible class
for $T(X,Y)$ can be introduced as follows. An operator $T \in B(X,Y)$ is said to be $R$-strictly singular if for any operator $L$ into $X$ such that $TL$ is tauberian, $L$ is weakly compact [3]. The perturbation class $P(T(X,Y))$ is not well known in general. However, for $X = L_1(\mu)$, we find that $P(T(L_1(\mu),Y))$ coincides with the class $Ro(L_1(\mu),Y)$ of all weakly precompact operators. Recall that $T \in B(X,Y)$ is said to be a weakly precompact operator if for every bounded sequence $(x_n) \subset X$, $(Tx_n)$ contains a weakly Cauchy subsequence.

**Proposition 11.** Let $Y$ be a Banach space such that $T(L_1(\mu),Y) \neq \emptyset$. An operator $K \in B(L_1(\mu),Y)$ is weakly precompact if and only if for every $T \in T(L_1(\mu),Y)$, the operator $T + K$ is tauberian.

Herman [6] call an operator $T \in B(X,Y)$ almost weakly compact if given a closed subspace $H \subset X$ such that $T|_H$ is an isomorphism, one has that $H$ is reflexive.

**Proposition 12.** For $T \in B(L_1(\mu),Y)$, the following statements are equivalent:

1. $T$ is weakly precompact;
2. $T$ is $R$-strictly singular;
3. $T$ is almost weakly compact.

**References**


