EXTRACTA MATHEMATICAE Vol. **10**, Núm. 1, 1 – 20 (1995)

Compact Weighted Composition Operators on Spaces of Continuous Functions: A Survey[†]

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(Presented by José G. Llavona)

AMS *Subject Class.* (1991): 47B38, 47B07, 46E40 *Received October* 17, 1994

The topological vector spaces of continuous functions and operators on them have been the objects of intensive and extensive study for the last several decades in the realm of topological algebraic analysis and have played significant roles in unification and classification of results in the broader areas of functional analysis. On any space of functions with some structures, there are two natural types of operators, one is the operator of multiplication and other is the operator of composition. When we combine them somehow we get another type of operator, known as the weighted composition operator, in short written as WCO. Among the operators the compact operators are of special significance as they make contact with concrete situations in application oriented studies. In this article, we have endeavoured to present a survey on the works done on the compact WCOs on several spaces of the continuous functions, specially the weighted spaces of continuous functions. This survey article has the following sections:

1. Introduction
2. Spaces of continuous functions
3. Appearance of WCOs
4. WCOs on weighted spaces
5. Compact WCOs

1. Introduction

For any two nonempty sets X and Y, and any non-trivial topological vector space E over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $F(X,E)$ and $F(Y,E)$ denote vector spaces

[†]Partially supported by U.G.C. Grant No. F8-7/91(RBB-II) and NBHM(DAE) Grant No. 40/16/93-G.

(over $\mathbb{K}$) of E-valued functions on X and Y respectively with pointwise linear operations. Let $B(E)$ denote the space of all operators on E (i.e. continuous linear transformations from E to itself). Let π be a function on Y which is scalar-valued, $B(E)$-valued, or E-valued in case E is an algebra, and let $T\colon Y \to X$ be a mapping such that $\pi \cdot f \circ T \in F(Y,E)$ whenever $f \in F(X,E)$, where the multiplication of π and the composite function $f \circ T$ is defined pointwise on Y. Then the map $f \to \pi \cdot f \circ T$ is a linear transformation from $F(X,E)$ to $F(Y,E)$. We denote it by $W_{\pi,T}$ and call it the weighted composition transformation from $F(X,E)$ to $F(Y,E)$ induced by the pair (π,T). In case both $F(X,E)$ and $F(Y,E)$ are topological vector spaces and $W_{\pi,T}$ is also continuous, it is called the weighted composition operator (WCO) from $F(X,E)$ to $F(Y,E)$. These operators behave like contravariant functors on categories of function spaces.

In case $\pi(y) = 1$ or $\pi(y) = I$, the identity operator on E, or $\pi(y) = e$, the unit of multiplication in algebra E for all y in Y, we write C_T in place of $W_{\pi,T}$ and call it the composition operator induced by T. When $X = Y$ and $T\colon X \to X$ is the identity map, we denote the corresponding WCO simply by M_π and call it the multiplication operator on $F(X,E)$ induced by π.

If C_T is a composition operator from $F(X,E)$ to $F(Y,E)$ and if M_π is a multiplication operator on $F(Y,E)$, then $W_{\pi,T}(= M_\pi \circ C_T)$ is a WCO from $F(X,E)$ to $F(Y,E)$. But the converse statement is not true. For instance, let us consider the case when $\pi(y) = 0$ for all $y \in Y$. Then $W_{\pi,T}$ is the zero operator from $F(X,E)$ to $F(Y,E)$ and so it is continuous, even if $C_T\colon F(X,E) \to F(Y,E)$ may not be continuous.

Thus the class of WCOs includes the two well known classes of operators, namely the class of composition operators and the class of multiplication operators. The study of WCOs has been the subject matter of several papers in recent years, see for example Nordgren [35], Cowen [11], Latushkin and Stëpin [28], and Singh and Manhas [47]. The initial study of WCOs was concentrated on L^p-spaces and H^p-spaces, which plays a very important role in the study of operators on Hilbert spaces, classical mechanics, statistical mechanics and ergodic theory (for instance, see Abrahamse [1], Hadwin and Hoover [15], Nordgren [36], Koopman [25], Von Neumann and Halmos [16, 17], Mayer [33], Lambert [26], and Hoover et al [19]).

In the last decade the study of WCOs was initiated on spaces of continuous functions to have its interactions with topological dynamics (see, for example Singh and Summers [56], Singh and Manhas [45, 48]).

The study of WCOs has been mostly carried out under the three main situations. Under the first situation, the function spaces are taken to be Banach spaces of measurable functions on measure spaces X and Y, and $\pi\colon Y \to \mathbb{K}$ and $T\colon Y \to X$ are measurable functions. WCOs on L^p-spaces fall under this category. Under the second situation, the function spaces are functional Banach spaces and the inducing functions have properties consistent with the concerned function spaces. For example, WCOs on Hardy spaces and Bergman spaces induced by holomorphic mappings have been studied under this setting. Under the third situation, we have spaces of continuous functions on topological spaces X and Y, and the inducing functions are continuous.

Our interest in this article centers around the compact WCOs on weighted locally convex spaces of vector-valued continuous functions (defined in section 2) which fall under the third situation mentioned above.

2. Spaces of Continuous Functions

Let X be a completely regular Hausdorff space and E a Hausdorff locally convex topological vector space (briefly, written as LCS) over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then by $C(X, E)$ we denote the vector space of all continuous functions from X into E, and by $cs(E)$ we mean the collection of all continuous seminorms on E. A function $f\colon X \to E$ is said to vanish at infinity if for each neighbourhood N of origin in E there exists a compact subset K of X such that $f(x) \in N$ for all x in $X \setminus K$, the complement of the set K in X, or equivalently, if the set $\{x \in X : p(f(x)) \geq \epsilon\}$ is relatively compact for every $p \in cs(E)$ and $\epsilon > 0$. Then we define

$$C_0(X, E) = \{f \in C(X, E) : f \text{ vanishes at infinity on } X\},$$
$$C_p(X, E) = \{f \in C(X, E) : f(X) \text{ is precompact in } E\}, \text{ and}$$
$$C_b(X, E) = \{f \in C(X, E) : f(X) \text{ is bounded in } E\}.$$

Clearly $C_0(X, E) \subset C_p(X, E) \subset C_b(X, E)$. When $E = \mathbb{K}$ with the usual topology, these spaces are respectively denoted by $C(X)$, $C_0(X)$, $C_p(X)$ and $C_b(X)$.

In case $X = \mathbb{N}$, the set of all natural numbers with the discrete topology, $C_b(\mathbb{N}) = \ell^\infty$, the Banach algebra of all bounded sequences in $\mathbb{K}$, and $C_0(\mathbb{N}) = c_0$, the Banach space of null sequences in $\mathbb{K}$. For f in $C(X)$ and t in E, the function f_t given by $f_t(x) = f(x) \cdot t$ for all x in X belongs to $C(X, E)$. If $f \in C_i(X)$, then $f_t \in C_i(X, E)$ for all t in E, where $i \in \{0, p, b\}$. In particular, for $f = 1$, the constant one function on X, $1_t \in C_i(X, E)$, where $i \in \{p, b\}$.

A real-valued function f on X is called upper–semicontinuous if the set

$\{x \in X : f(x) < a\}$ is open for all a in $\mathbb{R}$. By a weight we mean a nonnegative upper–semicontinuous function on X. Let V denote a family of weights on X. Then we say that $V > 0$ if for every $x \in X$ there is some $v_x \in V$ such that $v_x(x) > 0$; and that V is direct upward (or a Nachbin family [57]) if for every pair $u, v \in V$ and every $\alpha > 0$ there exists a $w \in V$ such that $\alpha u(x) \leq w(x)$ and $\alpha v(x) \leq w(x)$ for all x in X. Since there is no loss of generality, we hereafter assume that the sets of weights are directed upward. Now by a system of weights we mean a set V of weights on X which additionally satisfies that $V > 0$.

Let us now consider the following vector spaces (over $\mathbb{K}$) of continuous functions from X into E for a given system V of weights on X:

$$CV_0(X,E) = \{f \in C(X,E) : vf \text{ vanishes at infinity on } X \text{ for all } v \in V\},$$
$$CV_p(X,E) = \{f \in C(X,E) : vf(X) \text{ is precompact in } E \text{ for all } v \in V\},$$
$$CV_b(X,E) = \{f \in C(X,E) : vf(X) \text{ is bounded in } E \text{ for all } v \in V\}.$$

It is clear that $CV_0(X,E) \subset CV_p(X,E) \subset CV_b(X,E)$. If for each $v \in V$, $q \in cs(E)$ and $f \in C(X,E)$, we define

$$\|f\|_{v,q} = \sup\{v(x)q(f(x)) : x \in X\},$$

then $\|\cdot\|_{v,q}$ can be regarded as a seminorm on $CV_0(X,E)$, $CV_b(X,E)$ or $CV_p(X,E)$. We assume that each of these three spaces is equipped with the Hausdorff locally convex topology generated by the family $\{\|\cdot\|_{v,q} : v \in V, q \in cs(E)\}$ of seminorms. This topology is known as the weighted topology and has a basis of the closed absolutely convex neighbourhoods of origin of the form

$$B_{v,q} = \{f \in CV_0(X,E) : \|f\|_{v,q} \leq 1\}.$$

Thus $CV_0(X,E)$, $CV_p(X,E)$ and $CV_b(X,E)$ equipped with this weighted topo logy are weighted (locally convex) spaces of vector-valued continuous functions. In case $E = \mathbb{K}$ with the usual topology, it is convenient to write these spaces as $CV_0(X)$, $CV_p(X)$ and $CV_b(X)$ respectively; also we then write $\|\cdot\|_v$ in place of $\|\cdot\|_{v,q}$ for all $v \in V$, where $q(z) = |z|$, $z \in \mathbb{K}$. Further, if $E = (E,q)$ is any normed linear space and $v \in V$, we write $\|\cdot\|_{v,q} = \|\cdot\|_v$, and denote the corresponding ball $B_{v,q}$ simply by B_v.

If U and V are two systems of weights on X, then we write $U \leq V$ whenever for given $u \in U$ there is some $v \in V$ such that $u(x) \leq v(x)$ for all x in X. In this case, we have $CV_0(X,E) \subset CU_0(X,E)$, $CV_p(X,E) \subset CU_p(X,E)$ and $CV_b(X,E) \subset CU_b(X,E)$ as well as the inclusion map is continuous in all the three cases. If $U \leq V$ and $V \leq U$, then we say that U and V are equivalent systems of weights on X.

The spaces $CV_0(X)$ and $CV_b(X)$ were first introduced by Nachbin [34], and the vector-valued spaces $CV_0(X,E)$, $CV_p(X,E)$ and $CV_b(X,E)$ were studied in detail by Bierstedt [4] and Prolla [38]. Most of the commonly encountered spaces of continuous functions in analysis ([5, 61, 62, 41]) are the weigthted spaces as is evident from the following examples.

Let $\aleph_F$ denote the characteristic function of a subset F of X, and let us distinguish the following systems of weights on X:

$$\mathbf{1} = \mathbf{1}(X) = \{\alpha\aleph_X : \alpha > 0\},$$

$$\mathbf{K} = \mathbf{K}(X) = \{\alpha\aleph_K : \alpha > 0, K \subset X, K \text{ compact}\},$$

and $\mathcal{U} = \mathcal{U}(X)$ is the family consisting of all weights vanishing at infinity on X.

EXAMPLE 2.1. (a) $C\mathbf{1}_0(X,E) = (C_0(X,E), \mathrm{u})$, $C\mathbf{1}_p(X,E) = (C_p(X,E), \mathrm{u})$, and $C\mathbf{1}_b(X,E) = (C_b(X,E), \mathrm{u})$, where u in each case denote the topology of uniform convergence on X.
(b) $C\mathbf{K}_0(X,E) = C\mathbf{K}_p(X,E) = C\mathbf{K}_b(X,E) = (C(X,E), \text{c-op})$ where c-op denotes tha compact open topology.
(c) $C\mathcal{U}_0(X,E) = C\mathcal{U}_p(X,E) = C\mathcal{U}_b(X,E) = (C_b(X,E), \beta_0)$, where β_0 denotes the substrict topology.

For more information on weighted spaces of continuous functions, we refer to Summers [57] and Ruess and Summers [39].

3. Appearances of WCOs

As far as we know, the earliest appearance of a composition transformation in the literature was known in 1871 from a paper of Schroëder [40] in which for a given mapping T, a function f and a number α are to be found such that $f \circ T(z) = \alpha f(z)$ for all z in some domain. A solution to this has been given by Köenigs [24] in 1884 for the case when domain is the unit open disk and T is an analytic map.

The appearance of WCOs in the literature is in the classical work of Banach and Stone (see, [10]) while characterizing surjective isometries between spaces of continuous functions. The result says that, for compact Hausdorff spaces X and Y, if $A\colon C(X) \to C(Y)$ is a surjective isometry, then $A = W_{\pi,T}$, where $T: Y \to X$ is a homeomorphism and $\pi \in C(Y)$ with $|\pi(y)| = 1$ for all y in Y. The classical Banach-Stone Theorem has been extended in various directions (see, for example, Jerison [21], Cambern [6], Lau [29], Behrends [3], and Pathak [37]).

WCOs also appeared in the work [2] of Banach on characterization of isometries of $L^p[0,1]$, where $1 \leq p < \infty$, $p \neq 2$. Lamperti [27] generalizes these results of Banach to the spaces $L^p(X)$ for any σ-finite measure space X, which have been further generalized by Cambern [7] to the vector valued setting.

Isometries of H^p-spaces are WCOs (see, for example, Hoffman [18], Forelli [14], Cambern and Jarosz [8] and Lin [30]).

An operator A on a function module has the disjoint support property if and only if $A = W_{\pi,T}$ for some π and T (see [9]). A special case of this is a result of Jamison and Rajagopalan [20]. Using the theory of WCOs, Feldman and Porter [13] and Kitover [23] respectively studied lattice homomorphisms and d-homomorphisms between Banach lattices. Recently, Latushkin and Stëpin [28] made use of these operators in the study of linear extensions of dynamical systems, the theory of C^*-algebras and the theory of differential equations.

4. WCOs on Weighted Spaces

In order to avoid some minor problems and to have a healthier development of the theory, we shall require the following modest conditions on ingredients generating the weighted spaces of continuous functions:

(4.a) X is a completely regular Hausdorff space.

(4.b) V is a system of weights on X.

(4.c) E is a non-trivial LCS.

(4.d) Corresponding to each $x \in X$, there is an $f_x \in CV_0(X)$ such that $f_x(x) \neq 0$.

In case X happens to be locally compact, (4.d) is automatically satisfied. Since product of two weights is again a weight (cf. [57]), it follows that for any $\pi \in C(X)$ [or $\pi \in C(X,E)$ and $0 \neq p \in cs(E)$], the set

$$V \cdot |\pi| = \{v \cdot |\pi| : v \in V\} \quad [\text{or } V \cdot p \circ \pi = \{v \cdot p \circ \pi : v \in V\}]$$

is a Nachbin family on X. Further if π is nonzero at every point of X, $V \cdot |\pi|$ [or $V \cdot p \circ \pi$] is a system of weights on X. Again, if T is a continuous selfmap on X, then the set $V \circ T = \{v \circ T : v \in V\}$ is a system of weights on X.

A detailed study of composition operators on the weighted spaces of scalar-valued continuous functions has been made by Singh and Summers [56], and these results are carried over to the weighted spaces of vector-valued continuous functions in [46]. When π is a scalar or vector-valued function on X, the characterizations of multiplication operators and WCOs on $CV_i(X)$ and $CV_i(X, E)$, where $\{i \in 0, b\}$, have been obtained by Singh and Manhas ([43, 48]), and the authors ([51, 54]). For the operator-valued weights, the characterizations are given by Singh and Manhas ([44, 49]) and the authors ([52, 53]).

Before presenting these results we would like to mention here that the proof of all these characterizations uses more or less the same technique as presented in [56].

WCOs ON $CV_b(X)$ AND $CV_0(X)$. Before giving characterization of WCOs on $CV_b(X)$ we first consider the following proposition, a proof of which follows from [56, Lemma 2.1] by taking $v \cdot |\pi|$ in place of v.

PROPOSITION 4.1. *Suppose $\pi \in C(X)$ and T is a continuous selfmap on X. If $W_{\pi,T}\colon CV_0(X) \to CV_b(X)$ is continuous, then $V \cdot |\pi| \leq V \circ T$.*

THEOREM 4.2. *Let $\pi \in C(X)$ and T be a continuous selfmap on X. Then $W_{\pi,T}$ is a WCO on $CV_b(X)$ if and only if $V \cdot |\pi| \leq V \circ T$.*

Proof. We only prove the sufficient part as the necessary part follows from Proposition 4.1 above. If $V \cdot |\pi| \leq V \circ T$, then for given $v \in V$, there exists $u \in V$ such that $v(x)|\pi(x)| \leq u \circ T(x)$ for all x in X. Now, for all f in $CV_b(X)$, we have

$$\|W_{\pi,T} f\|_v = \|\pi \cdot f \circ T\|_v \leq \|f \circ T\|_{u \circ T} \leq \|f\|_u < \infty,$$

which is enough to conclude that $W_{\pi,T}$ is a WCO on $CV_b(X)$. ∎

The condition that $V \cdot |\pi| \leq V \circ T$ is necessary and sufficient for $W_{\pi,T}$ to be a WCO on $CV_b(X)$ as we see in Theorem 4.2 above, but in case of $CV_0(X)$ this condition is not sufficient though it is necessary as the following example shows:

EXAMPLE 4.3. Let $X = \mathbb{N}$ with the discrete topology, $V = \mathbf{1}(X)$, and suppose π and T are constant maps given by $\pi(n) = 1$ and $T(n) = 3$ for all n in $\mathbb{N}$. Then $V \cdot |\pi| \leq V \circ T$ but $W_{\pi,T}$ is not a WCO on $CV_0(\mathbb{N})$ $(= c_0)$. It is not even an into map, for if $f(n) = 1/n$ then $f \in c_0$ but $W_{\pi,T} f \notin c_0$.

Thus we require some more conditions on the pair (π, T) so that $W_{\pi,T}$ is a WCO on $CV_0(X)$. In order to present such a condition in the next theorem, we need the following definition:

If u is any weight on X and $\epsilon > 0$, then define $N(u, \epsilon) = \{x \in X : u(x) \geq \epsilon\}$ which is a closed subset of X because u is upper-semicontinuous.

THEOREM 4.4. *Let $\pi \in C(X)$ and T be a continuous selfmap on X. Then $W_{\pi,T}$ is a WCO on $CV_0(X)$ if and only if* (i) *$V \cdot |\pi| \leq V \circ T$, and* (ii) *for each $v \in V$, $\epsilon > 0$ and compact subset K of X, $T^{-1}(K) \cap N(v \cdot |\pi|, \epsilon)$ is a compact subset of X.*

Proof. It follows from Theorem 2.3 of [56] by replacing v by $v \cdot |\pi|$. ∎

WCOs ON $CV_b(X, E)$ AND $CV_0(X, E)$. Here we have three cases according to the range of function π, which may be contained in $\mathbb{K}$, E or $B(E)$. Let us consider the case when π is an operator-valued function on X. We use $B_b(E)$ to denote the space $B(E)$ of all operators on the LCS E equipped with the topology of uniform convergence on bounded subsets of E. This locally convex topology on $B(E)$ is induced by the family $\{\|\cdot\|_{p,K} : p \in cs(E), K \subset E, K \text{ bounded}\}$ of seminorms, where

$$\|A\|_{p,K} = \sup\{p(A(t)) : t \in K\}, \quad A \in B(E).$$

THEOREM 4.5. *Let $\pi \in C(X, B_b(E))$ and T be a continuous selfmap on X, and assume that $\pi(X)$ is equicontinuous. Then $W_{\pi,T}$ is a WCO on $CV_b(X, E)$ if and only if for each $(v, p) \in V \times cs(E)$, there exists $(u, q) \in V \times cs(E)$ such that $v(x)p(\pi(x)t) \leq u \circ T(x)q(t)$ for all x in x and t in E.*

The condition of the above Theorem is not sufficient for $W_{\pi,T}$ to be a WCO on $CV_0(X, E)$ as we have already noted in Example 4.3. The condition under which the map $W_{\pi,T}$ is a WCO on $CV_0(X, E)$ is presented in the following theorem:

THEOREM 4.6. *Let $\pi \in C(X, B_b(E))$ and T be a continuous selfmap on X. Assume that X is also a $k_{\mathbb{R}}$-space. Then $W_{\pi,T}$ is a WCO on $CV_0(X, E)$ if and only if* (i) *for each $(v, p) \in V \times cs(E)$, there exists $(u, q) \in V \times cs(E)$ such that $v(x)p(\pi(x)t) \leq u \circ T(x)q(t)$ for all x in X and t in E, and* (ii) *for each $(v, p) \in V \times cs(E)$, $\epsilon > 0$, and compact subset K of X, $T^{-1}(K) \cap \{x : v(x)p(\pi(x)t) \geq \epsilon\}$ is a compact subset of X for all nonzero t in E (X is called a $k_{\mathbb{R}}$-space if a function $f : X \to \mathbb{R}$ is continuous when its restriction to every compact subset K of X is continuous).*

Now,we consider the case when π is a vector-valued function on X. Here we assume that E is a Hausdorff locally convex algebra (briefly, written as LCA).

THEOREM 4.7. *Assume that E is a LCA, and let $\pi \in C(X,E)$ and T be a continuous selfmap on X. Then $W_{\pi,T}$ is a WCO on $CV_b(X,E)$ if and only if for each $(v,p) \in V \times cs(E)$, there exists $(u,q) \in v \times cs(E)$ such that $v(x)p(\pi(x)t) \leq u \circ T(x)q(t)$ for all x in X and t in E.*

The condition of the above theorem is not sufficient for $W_{\pi,T}$ to be a WCO on $CV_0(X,E)$ (cf. Example 4.3), and a necessary and sufficient condition for a map $W_{\pi,T}$ to be a WCO on $CV_0(X,E)$ is presented in the following theorem:

THEOREM 4.8. *Assume that E is a unital LCA in which multiplication is jointly continuous, and let $\pi \in C(X,E)$ and T be a continuous selfmap on X. Then $W_{\pi,T}$ is a WCO on $CV_0(X,E)$ if and only if* (i) *for each $(v,p) \in V \times cs(E)$, there exists $(u,q) \in V \times cs(E)$ such that $v(x)p(\pi(x)t) \leq u \circ T(x)q(t)$ for all x in X and t in E, and* (ii) *for each $(v,p) \in V \times cs(E)$, $\epsilon > 0$ and compact subset K of X, $T^{-1}(K) \cap N(v \cdot p \circ \pi, \epsilon)$ is a compact subset of X.*

In the third case when π is a scalar-valued function on X, we can write the corresponding results characterizing WCOs on $CV_b(X,E)$ and $CV_0(X,E)$ from Theorem 4.7 and Theorem 4.8 respectively.

Some examples of WCOs on weighted spaces of continuous functions are presented to illustrate the theory. As already noted in section 1, $W_{\pi,T}$ is a WCO on $CV_i(X,E)$ whenever M_π and C_T are respectively multiplication operator and composition operator on $CV_i(X,E)$, where $i \in \{b,0\}$. But it would be interesting to observe that even if one of π or T does not induce the corresponding operator, the pair (π,T) may still induce a WCO. This we shall present in the following examples:

EXAMPLE 4.9. Let $X = \mathbb{N}$ with the discrete topology and $V = \{\alpha v : \alpha > 0\}$, where $v(n) = n$ for all n in $\mathbb{N}$. Define $\pi(n) = 1/n$ for all n in $\mathbb{N}$ and T as follows:

$$T(n) = \begin{cases} \sqrt{n} & \text{if } n \text{ is a perfect square} \\ n & \text{otherwise .} \end{cases}$$

Then one can easily check that $V \cdot |\pi| \leq V$, $V \not\leq V \circ T$. but $V \cdot |\pi| \leq V \circ T$. In view of Theorem 4.2, it follows that $W_{\pi,T}$ is a WCO on $CV_b(\mathbb{N})$ but C_T is not an operator on $CV_b(\mathbb{N})$.

EXAMPLE 4.10. With X and V as in the above Example 4.9, if we define $\pi(n) = n$ and $T(n) = n^3$ for all n in $\mathbb{N}$, then $V \cdot |\pi| \not\leq V$, $V \leq V \circ T$ and $V \cdot |\pi| \leq V \circ T$. Again in view of Theorem 4.2, M_π is not an operator on $CV_b(\mathbb{N})$ but $W_{\pi,T}$ is a WCO on $CV_b(\mathbb{N})$.

EXAMPLE 4.11. Let $X = (0, \infty)$ with the usual relative topology and let $V = \{\alpha v : \alpha > 0\}$ where $v(x) = 1/x$ for all x in X. Define $\pi(x) = x^2$ and $T(x) = 1/x$ for all x in X. Then $V \cdot |\pi| \not\leq V$ and $V \not\leq V \circ T$ so that π and T do not induce the corresponding operator on $CV_b(X)$, in view of Theorem 4.2. However, $V \cdot |\pi| \leq V \circ T$ holds and therefore $W_{\pi,T}$ is a WCO on $CV_b(X)$.

For more examples of WCOs on weighted spaces of continuous functions, we refer to [52].

5. COMPACT WCOs

The study of compact weighted endomorphisms of the Banach algebra $C(X)$ of continuous functions on a compact Hausdorff space X was initiated by Kamowitz [22] in 1981. If $\pi \in C(X)$ and T is a continuous selfmap on X, then $W_{\pi,T}$ is a WCO on $C(X)$ (cf. Theorem 4.2). Kamowitz proved a WCO $W_{\pi,T}$ on $C(X)$ is compact if and only if each connected component of $N(\pi) = \{x \in X : \pi(x) \neq 0\}$ is contained in some open set on which T is constant. Singh and Summers [55], while observing that, for a completely regular Hausdorff space X, Kamowitz's condition does not guarantee that a WCO on $C_b(X)$ is compact, extended his result in two ways: firstly, by improving the condition and secondly, by placing the result in the setting of vector-valued continuous functions. If $\pi \in C(X)$ and $\epsilon > 0$, then the subset $\{x \in X : |\pi(x)| \geq \epsilon\}$ of X is denoted by $N(\pi, \epsilon)$.

THEOREM 5.1. *Let X be a completely regular Hausdorff space, E a Banach space, T a continuous selfmap on X and $\pi \in C_b(X)$ such that π is not identically zero on X. Then*

(5.1.1) *the WCO $W_{\pi,T}$ on $C_p(X, E)$ is compact if and only if E is finite dimensional and $T(N(\pi, \epsilon))$ is finite for every $\epsilon > 0$, and*

(5.1.2) *$W_{\pi,T}$ is a weakly compact operator on $C_p(X, E)$ if and only if E is reflexive and $T(N(\pi, \epsilon))$ is finite for every $\epsilon > 0$.*

Feldman [12] characterized compact WCOs on more general function spaces called Banach F-lattices, which also includes L^p-spaces and Banach lattices.

In a particular setting his result yields 5.1.1 for $C_b(X)$. An analogue of 5.1.1 for unweighted composition operators on $C_b(X, E)$ with a different proof has been presented in [42] in form of the following theorem:

THEOREM 5.2. *Let X be a completely regular Hausdorff space, E a Banach space and assume that T is a continuous selfmap on X. Then the composition operator C_T on $C_b(X, E)$ is compact if and only if E is finite dimensional and $T(X)$ is finite.*

The following corollary is an immediate consequence of this theorem.

COROLLARY 5.3. *If E is infinite dimensional Banach space, then no composition operator on $C_b(X, E)$ is compact.*

For a compact Hausdorff space X and a Banach space E, compact WCOs on $C(X, E)$ have been studied by Jamison and Rajagopalan in [20]. They have extended the result of Kamowitz [22] by carrying it over to $C(X, E)$ with π taking values in $B(E)$, and by weakening his hypothesis on the selfmap T by not requiring it to be continuous everywhere which is a part of the hypothesis in the paper of Kamowitz [22]. The following example illustrates this latter fact:

EXAMPLE 5.4. Let $X = [0, 1]$ with the usual relative topology and $E = \mathbb{K}$. Define $\pi(x) = 0$ for all $x \in X$ and T as follows:

$$T(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ 0 & \text{otherwise.} \end{cases}$$

Then the corresponding operator $W_{\pi,T}$ is continuous (even compact) but T is discontinuous at every point of X.

THEOREM 5.5. ([20]) *Let $T\colon X \to X$ and $\pi : X \to B(E)$ be mappings such that $W_{\pi,T}$ is a WCO on $C(X, E)$. Then $W_{\pi,T}$ is compact if and only if the following conditions hold:*

(5.5.1) *T is continuous on $N(\pi) = \{x \in X : \pi(x) \neq 0\}$.*

(5.5.2) *$\pi\colon X \to B(E)$ is continuous in the uniform operator topology.*

(5.5.3) *For every compact subset K of $N(\pi)$, $T(K)$ is finite.*

(5.5.4) *$\pi(x)$ is a compact operator on E for all x in X.*

(5.5.5) *For every bounded sequence* $\{f_n\}$ *in* $C(X, E)$ *and* $\epsilon > 0$, *there is a subsequence* $\{f_{n_k}\}$ *and a neighbourhood* G_ϵ *containing* Z *such that*

$$\|W_{\pi,T} f_{n_k}(x)\| < \epsilon \quad \text{for every } x \text{ in } G_\epsilon,$$

where $Z = \{x \in X : W_{\pi,T} f_n(x) = 0 \text{ for all } n\}$.

Let E^* denote the dual of E, and let A be a function algebra on X (i.e. a closed subalgebra of $C(X)$ which contains the constant functions and separates points of X). Then the space $A(X, E)$ defined as

$$A(X, E) = \{f \in C(X, E) : e^* \circ f \in A \text{ for all } e^* \in E^*\}$$

is a Banach space relative to the supremum norm of $C(X, E)$. An analogue of Theorem 5.5 characterizing compact WCOs on the space $A(X, E)$ has been obtained by Takagi in [59] which also includes results of his paper [58] in the function algebra setting. He proved that the condition 5.5.5 of Theorem 5.5 is removable and that no composition operator on $A(X, E)$ is compact when E is infinite dimensional. When $A = C(X)$, $A(X, E)$ is same as $C(X, E)$ and in this case Theorem A of [59] yields the following result without condition 5.5.5:

THEOREM 5.6. *Let* X *be a compact Hausdorff space,* E *a Banach space and suppose that* $W_{\pi,T}$ *is a WCO on* $C(X, E)$. *Then* $W_{\pi,T}$ *is compact if and only if the following conditions hold:*

(5.6.1) *for each connected component of* C *of* $N(\pi)$, *there exists an open set* G *containing* C *such that* T *is constant on* G.

(5.6.2) $\pi: X \to B(E)$ *is continuous in the uniform operator topology.*

(5.6.3) $\pi(x)$ *is a compact operator on* E *for all* x *in* X.

In case E is infinite dimensional, the identity operator I on E is noncompact and therefore the map $\pi(x) = I$ for all x in X does not satisfy condition (5.6.3) of Theorem 5.6 above. Hence we have

COROLLARY 5.7. *If* E *is infinite dimensional, then no composition operator on* $C(X, E)$ *is compact.*

In an another paper [60], Takagi and Wada have studied weakly compact WCOs on $A(X, E)$. Chan [9] has also worked in this direction: he proved that some of the conditions in Theorem 5.5 are redundant and gave a generalized version of Theorem 5.1. For a locally compact Hausdorff space X and a Banach space E, the following result of [9, Theorem 2.1] characterizes compact WCOs on $C_0(X, E)$:

THEOREM 5.8. *Let $W_{\pi,T}$ be an operator on $C_0(X,E)$. Then $W_{\pi,T}$ is compact if and only if the following conditions hold:*

(5.8.1) *Each $\pi(x)$ is a compact operator on E.*

(5.8.2) *$\pi\colon X \to B(E)$ is continuous in the uniform operator topology and the scalar function $x \to \|\pi(x)\|$ vanishes at infinity on X.*

(5.8.3) *T is locally constant on $N(\pi) = \{x \in X : \pi(x) \neq 0\}$.*

Remark. Condition (5.8.3) is equivalent to condition (5.5.3). As denoted earlier, let us put, for an operator-valued function π on X, $N(\pi,\epsilon) = \{x \in X : \|\pi(x)\| \geq \epsilon\}$ for every $\epsilon > 0$. When X is compact, Singh and Summers condition that "$T(N(\pi,\epsilon))$ is finite for every $\epsilon > 0$" is equivalent to (5.8.3) above.

COROLLARY 5.9. *Let $W_{\pi,T}$ be a WCO on $C(X,E)$. Then $W_{\pi,T}$ is compact if and only if the following conditions hold:*

(5.9.1) *Each $\pi(x)$ is a compact operator on E.*

(5.9.2) *$\pi\colon X \to B(E)$ is continuous in the uniform operator topology.*

(5.9.3) *$T(N(\pi,\epsilon))$ is finite for every $\epsilon > 0$.*

COROLLARY 5.10. ([9]) *Let X be a completely regular Hausdorff space, a Banach space, and suppose that $W_{\pi,T}$ is a WCO on $C_p(X,E)$. Then $W_{\pi,T}$ is weakly compact if and only if the following conditions hold:*

(5.10.1) *Each $\pi(x)$ is a weakly compact operator on E.*

(5.10.2) *$T(N(\pi,\epsilon))$ is finite for every $\epsilon > 0$.*

Remark. Corollary 5.9 and Corollary 5.10 are generalized versions of 5.1.1 and 5.1.2 respectively.

In function algebra setting, Lindström and Llavona [31] have obtained results about compact and weakly compact WCOs on the locally convex algebra $(C(X), \text{c-op})$.

PROPOSITION 5.11. ([31]) *Let X and Y be completely regular Hausdorff space, where Y is also a $k_{\mathbb{R}}$-space. Then the composition operator $C_T\colon (C(X), \text{c-op}) \to (C(Y), \text{c-op})$ is compact if and only if the inducing map $T\colon Y \to X$ is locally constant.*

THEOREM 5.12. ([31]) *Let X and Y be completely regular Hausdorff spaces, where Y is also a $k_{\mathbb{R}}$-space, and suppose that $C_T\colon (C(X), \text{c-op}) \to (C(Y), \text{c-op})$ is a composition operator. Then the following are equivalent:*

(5.12.1) *C_T is compact.*

(5.12.2) *C_T is weakly compact.*

(5.12.3) *T is locally constant.*

(5.12.4) *For every compact subset K of Y, $T(K)$ is a finite subset of X.*

Let us now consider compactness and weak-compactness of WCO on weighted spaces of vector-valued continuous functions. In [50], Manhas and the authors have studied compact composition operators on weighted spaces $CV_i(X, E)$, where $i \in \{b, 0\}$. The following results also from [50] are needed in presentation of this characterization:

PROPOSITION 5.13. *Let V be any system of weights on a completely regular Hausdorff space X and E be a LCS. Then the following statements are equivalent:*

(5.13.1) *Each $v \in V$ is bounded (respectively, vanishes at infinity) on X.*

(5.13.2) *For every $t \in E$, $1_t \in CV_b(X, E)$ (respectively, $CV_0(X, E)$).*

(5.13.3) *Every constant selfmap T on X induces a composition operator on $CV_b(X, E)$ (respectively $CV_0(X, E)$).*

LEMMA 5.14. *Let V be a system of weights on X and let $x_0 \in X$. Then there exists an open set containing x_0 on which each $v \in V$ is bounded.*

The following result [50, Theorem 3.2] shows that collection of compact composition operators on weighted spaces is not too large if the underlying space X is connected and the space E is finite dimensional.

THEOREM 5.15. *Let V be a system of weights on X satisfying the condition 5.13.1. Also assume that X is connected and E is a finite dimensional LCS. Then a composition operator C_T on $CV_b(X, E)$ (or $CV_0(X, E)$) is compact if and only if T is constant.*

Proof. We only give the proof of direct part. If $x_1, x_2 \in X$ such that $y_1 = T(x_1) \neq T(x_2) = y_1$ then by Lemma 5.14 there exists an open set G containing y_1 on which each $v \in V$ is bounded and $y_2 \notin G$. Consider a continuous function $f\colon X \to [0,1]$ such that $f(y_1) = 1$ and $f(x) = 0$ for all $x \notin G$. Choose $q \in cs(E)$ and $t \in E$ such that $q(t) \neq 0$ and define $f_t \in CV_b(X,E)$ as $f_t(x) = f(x)t/q(t)$ for all $x \in X$. Also write $g_n(x) = f^{n-1}(x)f_t(x)$ for all $n \geq 1$, where $f^0(x) = 1$ and $f^{n-1}(x) = f(x)f^{n-2}(x)$ for all $x \in X$ and $n \geq 2$. Then $F = \{g_n : n \geq 1\}$ is a bounded subset of $CV_b(X,E)$ and so if C_T is compact, we have a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and a function $g \in CV_b(X,E)$ such that $\{C_T g_{n_k}\}$ converges to g. From this, we conclude that $q \circ g_{n_k}(T(x))$ converges to $q \circ g(x)$ for all x in X, which implies that $q \circ g(x)$ is a characteristic function with $q \circ g(x_1) = 1$ and $q \circ g(x_2) = 0$. This is impossible since X is connected and $q \circ g$ is continuous on X. Thus we must have $T(x_1) = T(x_2)$ for all $x_1, x_2 \in X$, showing that T is a constant map. ∎

Remark. (1) This theorem is a generalization of Theorem 5.11.

(2) In case E is infinite dimensional, the constant map may not induce compact composition operators as the following example shows:

EXAMPLE 5.16. Let $X = [0,1]$ with the usual relative topology, $V = \mathbf{1}(X)$ and $E = c_0$. Then $CV_b(X,E)$ is a Banach space with the supremum norm. Let T be the constant map given by $T(x) = 1/3$ for all x in X. Then $C_T f(x) = \{f_n(1/3)\}$ for $f = \{f_n\}$ shows that C_T is not compact.

In a recent work [32] Manhas and Singh have studied compact and weakly compact WCOs on weighted spaces, from which follows the results of Chan [9] and some results of Lindström and Llavona [31].

Before giving Theorem 2.1 of [32] which characterizes compact WCOs on weighted spaces, we first require the generalized Arzela-Ascoli Theorem from [39] in the following form:

THEOREM 5.17. *Let X be a completely regular Hausdorff $k_{\mathbb{R}}$-space and E be a quasicomplete LCS. Then a subset H of*

$$(C_b(X,E), \beta_0) = C\mathcal{U}_0(X,E)$$

is relatively compact if and only the following conditions hold:

(5.17.1) *H is equicontinuous.*

(5.17.2) *$H(x) = \{h(x) : h \in H\}$ is relatively compact in E for all x in X.*

(5.17.3) *H is uniformly bounded.*

THEOREM 5.18. *Let X be a completely regular Hausdorff $k_{\mathbb{R}}$-space, E a quasicomplete LCS and $V = \mathcal{U}$, and assume that $\pi\colon X \to B(E)$ and $T\colon X \to X$ are continuous mappings such that $W_{\pi,T}$ is a WCO on $CV_0(X,E)$. Then $W_{\pi,T}$ is compact if and only if the following conditions hold:*

(5.18.1) *Each $\pi(x)$ is a compact operator on E.*

(5.18.2) *T is locally constant on $N(\pi)$.*

Proof. Suppose $W_{\pi,T}$ is a compact operator on $CV_0(X,E)$. Then using Theorem 5.17 it is easy to show (5.18.1). To prove (5.18.2), suppose that T is not locally constant on $N(\pi)$. Then there exists an element x_0 in $N(\pi)$ such that T is not constant on any open set containing x_0 in X. Let Δ be an open neighbourhood base at x_0 in X. Then there is a net $\{x_G : G \in \Delta\}$ in X such that $x_G \to x_0$ and $T(x_G) \neq T(x_0)$ for each $G \in \Delta$. Let $f_G \in C(X)$ such that $0 \leq f_G \leq 1$, $f_G(T(x_0)) = 1$ and $f_G(T(x_G)) = 0$, and choose $t \in E$ and $q \in cs(E)$ such that $q(\pi(x_0)t) \neq 0$. If we define $g_G(X) = f_G(x)t$ for all x in X, then $M = \{g_G : G \in \Delta\}$ is a bounded subset of $CV_0(X,E)$. But since

$$\sup\{q[W_{\pi,T}g_G(x_G) - W_{\pi,T}g_G(x_0)] : g \in \Delta\} = q(\pi(x_0)t) > 0,$$

for all $G \in \Delta$, we have a contradiction to the equicontinuity of $W_{\pi,T}(M)$.

Conversely, assume that the conditions (5.18.1) and (5.18.2) hold. Then using Theorem 5.17 it can be easily shown that $W_{\pi,T}$ is compact. ∎

Remark. Theorem 5.18 is a generalized version of 5.11.

THEOREM 5.19. ([32]) *Under the hypothesis of Theorem 5.18, a WCO $W_{\pi,T}$ on $CV_0(X,E)$ is weakly compact if and only if the following conditions hold:*

(5.19.1) *Each $\pi(x)$ is a weakly compact operator on E.*

(5.19.2) *T is locally constant on $N(\pi)$.*

COROLLARY 5.20. *Under the hypothesis of Theorem 5.18, the following conditions are equivalent when E is finite dimensional:*

(5.20.1) *$W_{\pi,T}$ is a compact operator on $CV_0(X,E)$.*

(5.20.2) *$W_{\pi,T}$ is a weakly compact operator on $CV_0(X,E)$.*

(5.20.3) *T is locally constant on $N(\pi)$.*

Remark. (i) Theorem 5.19 is a generalization of Theorem 3.1 of [9] and Corollary 5.10.

(ii) In a particular setting Corollary 5.20 reduces to Theorem 5.12 and Theorem 5.15.

Now let us consider the following example from [32] which certifies the distinction from the special situations considered earlier.

EXAMPLE 5.21. Let $X = \mathbb{N}$, with the discrete topology, and $E = (C(\mathbb{R})$, c-op). Define $T\colon \mathbb{N} \to \mathbb{N}$ as

$$T(n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n+1 & \text{otherwise,} \end{cases}$$

and $\pi\colon X \to B(E)$ as $\pi(n) = n^2 A_n$ for all n in $\mathbb{N}$, where $A_n\colon E \to E$ is given by $A_n f(t) = f(n)$ for all $t \in \mathbb{R}$ and for all $f \in E$. Then clearly T is locally constant and each $\pi(n)$ is a compact operator on E. Let $V = \mathbf{K}(\mathbb{N})$. Then clearly π and T induce the compact (weakly compact) WCO $W_{\pi,T}$ on $CV_0(X,E)$. On the other hand if we take $V = \{\alpha v : \alpha > 0\}$, where $v(n) = 1/n$ for all $n \in \mathbb{N}$, then π and T do not induce the compact (weakly compact) WCO $W_{\pi,T}$ on $CV_0(X,E)$. Moreover, in this case $W_{\pi,T}$ is not even an operator on $CV_0(X,E)$. Thus we have seen that $\pi\colon X \to B(E)$ is a continuous operator-valued mappings such that each $\pi(x)$ is a compact (weakly compact) operator on E and $T\colon X \to X$ is locally constant, even then $W_{\pi,T}$ fails to be a compact WCO on $CV_0(X,E)$.

This example illustrates the distinction from the situation considered in ([9, 31, 59], and it further shows that the compactness (weak compactness) of a WCO is very much influenced by the behaviour of these three variables,namely, V, π and T.

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