EXTRACTA MATHEMATICAE Vol. 10, Núm. 1, 27 – 36 (1995)

# On Banach Spaces $X$ such that $L(L_p, X) = K(L_p, X)$

Jesús M.F. Castillo

*Dpto. de Matemáticas, Univ. de Extremadura,*
*Avda. de Elvas, s/n,* 06071-*Badajoz, Spain*

AMS *Subject Class.* (1991): 46B20, 47B10, 46B25

*Received July* 1, 1995

## 1. Introduction

Several results in Banach spaces theory contain the hypothesis $L(\ell_p, X) = K(\ell_p, X)$. In this paper we prove that

(*) $L(\ell_p, X) = K(\ell_p, X) \Longleftrightarrow$ weakly-$p^*$-summable sequences in $X$ converge

something which is essentially known, but for which the author was unable to find any explicit reference. That result suggests to define the ideal of $p$-converging operators, $C_p$, as those operators transforming weakly-$p$-summable sequences into norm null sequences. They receive its name from the fact that $C_1 = \mathfrak{U}$ (the ideal of unconditionally converging operator). The proof of (*) is based upon the following result: any bounded sequence in $\ell_p$ admits a weakly-$p^*$-converging (see below the definition) subsequence. That suggests to define the ideal of weakly-$p$-compact operators, $W_p$, in an obvious manner. To see the versatility of this approach, let us consider the main technical lemma of [1]: if $X$ is a Banach space such that, for some $p > 1$, every operator form $X$ into $\ell_p$ is compact, then, for each $r < p$, all operators from $X$ into $\ell_p$ are compact. This can be easily proved as follows: the hypothesis is $id(X^*) \in C_p$; since $C_p \subseteq C_r$ for all $r < p$, the conclusion follows.

Other applications are given: a) characterization $\{\ell_2\}$-strictly singular operators in $L_p$-spaces, $1 < p < 2$, completing in this way L. Weis results [16], and b) a criterion to find copies of $\ell_p$.

## 2. Results

Throughout the paper, $p$ and $p^*$ are always conjugate numbers: $1/p + 1/p^* = 1$. When $p = 1$ its conjugate number is $+\infty$ and, whenever it occurs, $\ell_\infty$ has to be understood as $c_0$.

A sequence $(x_n)$ in a Banach space $X$ is said to be weakly-$p$-summable, $1 \leq p < +\infty$, when for each $f \in X^*$, $\sum_n |f(x_n)|^p < +\infty$; equivalently, there is a $C > 0$ such that for any $(t_n) \in \ell_{p^*}$ .

$$\sup \left\| \sum_{k=1}^{k=n} t_k x_k \right\| \leq C \|(t_n)\|_{p^*}.$$

We shall say that a sequence $(x_n)$ is weakly-$p$-convergent to a point $x$, if the sequence $(x_n - x)$ is weakly-$p$-summable. We admit that weakly-$\infty$-convergent sequences are the weakly convergent sequences.

We will work mainly with the following operator ideals: $K$ (compact operators), $W$ (weakly compact operators), $B$ (completely continuous operators, i.e., those sending weakly convergent sequences into norm converging sequences), $\mathfrak{U}$ (unconditionally summing operators, i.e., those sending weakly-1-summable sequences into unconditionally summable sequences) and $\mathfrak{S}$ (strictly singular operators, i.e., the operators whose restriction to any infinite dimensional subspace is not an isomorphism). An operator is said to be $\mathcal{F}$-singular, where $\mathcal{F}$ denotes a given class of Banach spaces, if its restriction to any subspace of the class $\mathcal{F}$ is not an isomorphism. For instance, $\mathfrak{U} = \{c_0\}$-strictly singular operators. $\mathcal{F}$-strictly singular operators are not, in general, operator ideals.

Definition 1. We shall say that an operator $T \in L(X,Y)$ is $p$-converging, $1 \leq p < +\infty$, if it transforms weakly-$p$-summable sequences of $X$ into norm null sequences of $Y$. We shall denote this class $C_p$.

The scale of $C_p$ ideals is intermediate between the ideals $B$ and $\mathfrak{U}$, since we clearly have: $C_\infty = B$ and $C_1 = \mathfrak{U}$. Some elementary but useful properties of the ideals $C_p$ are (see [6]):

Lemma 2. *Let $1 < p < +\infty$. $C_p$ is an injective, non-surjective, closed operator ideal.*

Definition 3. Let $1 \leq p \leq +\infty$. A subset $K$ of a Banach space $X$ is said to be relatively weakly-$p$-compact if each bounded sequence admits a weakly-$p$-converging subsequence.

The following result is in [6, Prop. 1.4]. Since it is essential for our purpose, we reproduce it here.

Proposition 4. *Let $1 < p < +\infty$. The closed unit ball of $\ell_p$ is a weakly-$p^*$-compact set.*

*Proof.* Let $(x_n)$ be a bounded sequence in $\ell_p$. It admits a weakly convergent subsequence $(x_k)$. Let $x$ be its weak limit and let us call $y_k = x_k - x$. If $(y_k)$ is norm null, we have finished. If not, and we have $\|y_k\| \geq \epsilon$ for some subsequence, applying the Bessaga-Pelczyński selection principle we obtain a new subsequence, equivalent to the canonical basis $(e_n)$ of $\ell_p$, which is weakly-$p^*$-summable. ∎

COROLLARY 5. *An operator $T \in C_p(X, Y)$ if and only if for each $j \in L(\ell_p, X)$ the composition $T \circ j$ is compact.*

From this and Pitt's lemma (see [13]) we derive:

LEMMA 6. *Let $1 < r < +\infty$. For $p < r^*$, $C_p(\ell_r) = L(\ell_r)$. For $p \geq r^*$, $C_p(\ell_r) = K(\ell_r)$.*

DEFINITION 7. An operator $T \in L(X, Y)$ is said to be weakly-$p$-compact if the image of the unit ball of $X$ is a relatively weakly-$p$-compact set in $Y$. The class of weakly-$p$-compact operators will be denoted $W_p$.

Clearly, $W_p$ operators are meant to be gradations of the class of weakly compact operators. The well known duality between $B(= C_\infty)$ and $W(= W_\infty)$ expressed by $B \circ W = K$ still works for other $p$: $C_p \circ W_p = K$.

LEMMA 8. *For $1 \leq p < +\infty$, $W_p$ is an injective, surjective non-closed operator ideal. Moreover, $W_1^2 = K$.*

(See [6]).

*Remark.* The ideals $C_p$, $1 \leq p \leq +\infty$, and $W_1$ are not idempotent. The ideal $W_\infty$ is idempotent. We do not know whether $W_p$, $1 < p < +\infty$, is idempotent.

LEMMA 9. *If $T \in W_p$ then $T^* \in C_r$ for all $r < p^*$.*

*Proof.* We simply prove that for any operator $j \in L(\ell_{r^*}, Y^*)$ the composition $T^* \circ j$ is compact, $r^* > p$. This happens if and only if its dual $j^* \circ T^{**}$ is compact, which is obviously the case if $T^{**} \in W_p$. Since the unit ball of $X$ is $\sigma(X^{**}, X^*)$-dense in the unit ball of $X^{**}$, $T \in W_p$ if and only if $T^{**} \in W_p$, which finishes the proof. ∎

COROLLARY 10. *Let $1 < p < +\infty$. For $p < r^*$, $W_p(\ell_r) = K(\ell_r)$. For $p \geq r^*$, $W_p(\ell_r) = L(\ell_r)$.*

As an application of Orlicz's result (see [8]): Let $X$ be a Banach space of cotype $s$. If $(x_n)$ is a weakly-$p$-sequence of $X$ and $p < s$, then

$$\sum \|x_n\|^{\frac{ps^*}{s^*-p}} < +\infty.$$

It follows

THEOREM 11. *$id(L_r) \in C_p$ if and only if $p < \min(2, r^*)$.*

*Remark.* Notice that since $\ell_r$ is a subspace of $L_r$, we could deduce from this result the part a) of Lemma 6, but only for $r \geq 2$: we get that the identity of $\ell_r$ is also absolutely $(pr^*/(r^* - p), p)$ summing for $p < r^*$; but when $1 < r < 2$, with this method we only obtain that the identity of $\ell_r$ is absolutely $(2p/(2-p), p)$ for $1 \leq p < 2$. When $p \geq 2$ the summation becomes meaningless. Lemma 6 says that there is a residual effect extending up to $r^*$.

From now on we distinguish the cases $1 < r < 2$ and $r \geq 2$.

LEMMA 12. *Let $2 \leq r$. $id(L_r) \in W_2$.*

*Proof.* This is a consequence of the Kadec-Pelczyǹski characterization of basic sequences in $L_p$ spaces. Thus, if we take a bounded sequence in $L_r$ and extract a weakly convergent subsequence, then: if this is norm null, we have finished; if not, we apply the Kadec-Pelczyǹski alternative: the final subsequence will be, at worst, equivalent to the unit vector basis of $\ell_2$; that is weakly-2-summable. ∎

The distribution of $C_p$ and $W_p$ operators in $L_r$ spaces, $r \geq 2$, immediately follows:

THEOREM 13. *Let $2 \leq r$.*

(a) *For $p < r^*$, $C_p(L_r) = L(L_r)$, $W_p(L_r) = K(L_r)$.*

(b) *For $r^* \leq p < 2$, $C_p(L_r) = C_{r^*}(L_r)$, $W_p(L_r) = W_{r^*}(L_r)$.*

(c) *For $2 \leq p$, $C_p(L_r) = K(L_r)$, $W_p(L_r) = L(L_r)$.*

We see that the three classes are actually different: the identity of $L_r$ belongs to (a) but not to (b). Any projection $P : L_r \longrightarrow \ell_2$, considered as an operator $L_r \longrightarrow L_r$, belongs to (b) but is not compact.

When $1 < r < 2$, the result of Kadec and Pelczyǹski no longer applies. We will work by duality. In the literature, weakly-$p$-summable sequences have been called $p$-Hilbertian. A sequence $(x_n)$ such that

$$\sum_n t_n x_n \text{ converges } \Longrightarrow (t_n) \in \ell_p$$

is called $p$-Besselian. The following lemma is a straightforward:

LEMMA 14. *Let $(x_n)$ be a basis in a Banach space $X$, and let $(x_n^*)$ be the sequence of associated functionals. $(x_n)$ is p-Hilbertian (resp., p-Besselian) if and only if $(x_n^*)$ is p-Besselian (resp., p-Hilbertian).*

*Remark* 15. When $(x_n)$ is simply a basic sequence, and $(x_n^*)$ a sequence of biorthogonal functionals, the equivalences of Lemma 14 do not hold. It is still true that if $(x_n^*)$ is $p$-Hilbertian, then $(x_n)$ is $p$-Besselian. In the opposite direction, we have that when $[x_n^*]$ (the closed linear span of $(x_n^*)$) is complemented in $X^*$, then if $(x_n^*)$ is $p$-Besselian, then $(x_n)$ is $p$-Hilbertian.

LEMMA 16. *Let $1 < r < 2$. $id(L_r) \in W_{r^*}$.*

*Proof.* Let $(x_m)$ be a bounded sequence in $L_r$, $1 < r < 2$. A subsequence of $(x_m)$ is weakly convergent to a point $x$ due to the reflexivity of $L_r$. The sequence $(y_j) = (x_{m(j)} - x)$ is weakly null. If it is norm null, then there is nothing to prove. If not, assume that $\|y_j\| \geq \epsilon > 0$, and pick a basic subsequence $(y_k)$ out of it. Let $(y_k^*)$ be a bounded sequence of biorthogonal functionals. Since (possibly a subsequence) is weakly convergent, then we assume that is weakly null. Obviously $\limsup \|y_k^*\| > 0$. Applying then the result of Kadec-Pelczyǹski, it has a subsequence $(y_n^*)$ equivalent to the unit vector basis of $\ell_{r^*}$ or $\ell_2$. In either case, $(y_n^*)$ is $r^*$-Besselian. Since $[y_n^*]$ is complemented in $L_{r^*}$, $(y_n)$ is $r^*$-Hilbertian, that is, weakly-$r^*$-summable. ∎

We complete the distribution of $C_p$ and $W_p$ operators in $L_r$ spaces:

THEOREM 17. *Let $1 < r < 2$.*

(a) *For $p < 2$, $C_p(L_r) = L(L_r)$, $W_p(L_r) = K(L_r)$.*

(b) *For $2 \leq p < r^*$, $C_p(L_r) = C_2(L_r)$, $W_p(L_r) = W_2(L_r)$.*

(c) *For $p \geq r^*$, $C_p(L_r) = K(L_r)$, $W_p(L_r) = L(L_r)$.*

*The three classes are different. Projections $P : L_r \longrightarrow \ell_r$ belong to* (b) *and are not compact.*

*Remark* 18. It is well known that absolutely-$p$-summing operators are weakly compact. This follows from the Grothendieck-Pietsch domination theorem which implies that $\Pi_p$ operators subfactorize through an $L_p$ space.

Using Lemma 2 we obtain:

THEOREM 19. *Let* $1 \leq p < +\infty$. $\Pi_p \subseteq W_2$.

For $L_1(\mu)$-spaces, $\mu$ finite without atoms, we have that since $L_1$ spaces have cotype 2, $id(L_1) \in C_r$, for all $r < 2$. Then $W_r(L_1) = K(L_1)$ for all $r < 2$. Since $\ell_p$, $p \leq 2$, is a quotient and a subspace of $L_1$, the quotient operator $L_1 \longrightarrow \ell_p$ regarded as in $L(L_1)$ belongs to $W_{p^*}$ (optimum). This proves that the classes $W_r(L_1)$ are different for $r \geq 2$. Applying a result of Bourgain, one sees that for $r \geq 2$, any $C_r$ operator $T : L_1 \longrightarrow X$ is such that $T \circ i$ is compact, therefore $C_r(L_1) = B(L_1)$ for $r \geq 2$. Thus we have

PROPOSITION 20.

*For* $1 \leq p < 2, \quad C_p(L_1) = C_1(L_1), \quad W_p(L_1) = K(L_1)$.

*For* $2 \leq p, q \leq +\infty, \quad C_p(L_1) = B(L_1), \quad W_p(L_1) \neq W_q(L_1)$.

It is well known that for an operator $T : C(K) \longrightarrow X$ the following are equivalent: $T \in \mathfrak{U}$, $T \in W$ and $T \in B$. Therefore $\mathfrak{U} = W = B$. For $p \geq 2$ the natural injection $W_r(C(K))$ are different for $r \leq 2$. In [6], it is given an example showing that, in general, $W \neq W_p$ in $C(K)$-spaces. With a slight modification we can get $W_p$ operators from a $C(K)$ space which are not $W_r$ for $r < p$: consider a sequence $(\theta_n)$ in $\ell_p$ but not in $\ell_r$ for $r < p$, and repeat the construction above beginning with an $f$ such that $(\langle f, \theta_n r_n \rangle)_n \notin \ell_r$ for no $r < p$.

From Theorem 13 and 17 we derive:

PROPOSITION 21.

*Let* $2 \leq p < +\infty$. $L(L_p, X) = K(\ell_p, X)$ *if and only if* $id(X) \in C_2$.

*Let* $1 < p < 2$. $L(L_p, X) = K(L_p, X)$ *if and only if* $id(X) \in C_{p^*}$.

## 3. Applications

It is an open problem to find conditions on a Banach space in order to contain $\ell_p$. When $p = 1$ a solution is given by Rosenthal $\ell_1$ theorem. For $c_0$ ($p = +\infty$), $id(X) \notin C_1$ is a necessary and sufficient condition. It is therefore quite a tempting conjecture that $id(X) \notin C_{p^*}$ is, in some sense, the natural condition for $\ell_p$. The mutual situation of those ideals shows that this cannot be so unless we introduce at some point the information that $p^*$ is the limit number. Regarding Lemma 9, the next theorem does exactly this:

THEOREM 22. *Let $X$ be a Banach space. If $id(X) \notin C_{p^*}$ and $id(X^*) \in W_p$ then $X$ contains a copy of $\ell_p$.*

*Proof.* Since the identity of $X$ is not $p^*$-converging, it is possible to find a semi-normalized sequence $(x_n)$ in $X$ admitting an upper-$\ell_p$-estimate. We shall consider it a basic sequence henceforth. Since $X$ is reflexive we can find a weakly null biorthogonal sequence $(x_n^*)$ in $X^*$. Since $id(X^*) \in W_p$ there is a weakly-$p$-summable subsequence $(x_k^*)$, and this makes the corresponding subsequence $(x_k)$ admit a lower-$p$-estimate. ∎

COROLLARY 23. *Let $1 < p < 2$. Let $X$ be a closed subspace of $L_p(\mu)$. Then $X$ contains a copy of $\ell_p$ if and only if $X^*$ contains a copy of $\ell_{p^*}$.*

*Proof.* By [11, Theorem 13], if $id(X^*) \in C_p$ then $X$ does not contain $\ell_p$. Since $id(X) \in W_{p^*}$, if $id(X^*) \notin C_p$ we obtain a copy of $\ell_{p^*}$ inside $X^*$. ∎

COROLLARY 24. *Let $X$ be a closed subspace of $L_p(\mu)$, $1 < p < 2$. $id(X) \in C_2$ if and only if $X$ does not contain a copy of $\ell_2$.*

L. Weis proved in [3] that $\mathfrak{S}(L_p(\mu)) = \{\ell_2, \ell_p\}$-strictly singular operators. In addition to that, we can show:

THEOREM 25. *Let $1 < p < 2$. Let $T \in L(X, L_p(\mu))$ be an operator. $T$ does not belong to $C_2$ if and only if there is a subspace $Z$ of $X$, isomorphic to $\ell_2$, such that $T$ acts as an isomorphism when restricted to $Z$.*

*Proof.* Let $T : X \longrightarrow L_p(\mu)$ be a continuous operator not 2-converging. Then there is a sequence $(x_n)$ in $X$ weakly-2-summable such that, if we pose $y_n = Tx_n$, $\|y_n\| > \epsilon$. Obviously $\|x_n\| > \epsilon_1$. Now, $(y_n)$ has a subsequence equivalent to the unit vector basis of $\ell_2$. To see this, we pick a basic subsequence out of it and consider a sequence $(y_n^*)$ of associated functionals norm bounded and weakly null. By Kadec-Pelczyǹski criterion, it admits a subsequence equivalent to the unit vector basis of $\ell_{p^*}$ or $\ell_2$. It is, therefore, weakly-2-summable. The corresponding subsequence of $(y_n)$ admits a lower-2-estimate and it is, therefore, equivalent to the unit vector basis of $\ell_2$. Thus, if we consider the arrows: $\ell_2 \longrightarrow X \longrightarrow L_p(\mu)$ we obtain:

$$\|\sum \lambda_n y_n\| \leq \|T\| \, \|\sum \lambda_n y_n\| \leq \|T\| \, K(\sum y_n^2)^{\frac{1}{2}} \leq \|T\| \, KK' \|\sum \lambda_n y_n\|$$

This says that $T$ is an isomorphism restricted to the subspace, isomorphic to $\ell_2$, spanned by certain subsequence of $(x_n)$. ∎

COROLLARY 26. *$C_{(cotype\ L_p)^*}(X, L_p) = \{\ell_{cotype\ L_p}\}$-strictly singular operators.*

## 4. DUNFORD-PETTIS PROPERTY OF ORDER $p$

The DunfordPettis property is $W \subseteq C_\infty$. A gradation is the Dunford-Pettis property of order $p$, $(\mathrm{DPP}_p)$, $1 \leq p \leq +\infty$, namely $W \subseteq C_p$. Clearly, $\mathrm{DPP}_\infty$=DPP and every Banach space has $\mathrm{DPP}_1$.

These properties were considered in [5] and [7]. There is, however, a simple and direct proof for vector sequence spaces $\ell_p(X)$ that in some cases works better than that presented in [7].

To study Dunford-Pettis properties of vector valued sequence spaces our notation follows [2]: especially, if $T : \ell_p(X) \longrightarrow Y$ is an operator, it determines a unique sequence $(T_k)$ of operators, $T_k \in L(X, Y)$, such that $T(x) = \sum T_k(x_k)$, which is called its representing sequence. With some notational abuse we write $T = \sum T_k$, but the meaning is obvious.

The following result is essentially proved in [2]:

LEMMA 27. *Let $A \subseteq \ell_p(X)$. The following are equivalent:*

1. *For any operator $T : \ell_p(X) \longrightarrow Y$ with representing sequence $T_k$,*

$$\sum^N T_k \longrightarrow T \text{ uniformly on } A.$$

2. $\sup_{x \in A} \sum_N^\infty \|x(k)\|^p \underset{N}{\longrightarrow} 0.$

Now we can characterize the $\mathrm{DPP}_r$ in $\ell_p(X)$ in terms of that property in $\ell_p$ and $X$:

THEOREM 28. *If $X$ and $\ell_p$ have the Dunford-Pettis property of order $r$ then $\ell_p(X)$ has the same property.*

*Proof.* Let $T : \ell_p(X) \longrightarrow Y$ be a weakly compact operator with representing sequence $T_k$, that is, $T = \sum T_k$ pointwise. Let $(f_n)$ be a weakly-$r$-summable sequence in $\ell_p(X)$, $r < p^*$. We make our claim:

$$\sup_n \sum_N^\infty \|f_n(k)\|^p \underset{N}{\longrightarrow} 0.$$

If not, an $\epsilon > 0$ and a sequence $(N(i))$ of naturals exist such that:

$$\sum_{k=N_i}^{k=N_{i+1}} \|f_{n_i}(k)\|^p > \epsilon.$$

Let us determine normalized elements $x^*(k) \in X^*$ such that $\langle x_i^*(k), f_{n_i}(k)\rangle = \|f_{n_i}(k)\|$. If we call now $y^*(k) = x_i^*(k)$ for $N_i \leq k < N_{i+1}$, we have a bounded sequence in $X^*$, which defines an operator $\ell_p(X) \longrightarrow \ell_p$. It transforms $(f_n)$ into a weakly-$r$-summable sequence in $\ell_p$, which must be norm null. Thus we have:

$$\sup_n \left[\sum_{k=N}^\infty |\langle x^*(k), f_n(k)\rangle|^p\right]^{1/p} \underset{N}{\longrightarrow} 0.$$

This gives us a contradiction. Once the claim has been proved we use the Lemma to know that $\sum^N T_k$ converges uniformly to $T$ over $\{f_n\}$. Thus we get

$$T(\{f_n\}) \subseteq \sum^{N(\epsilon)} T_k(\{f_n\}) + \epsilon B_Y.$$

Since $X$ has DPP$_r$, the first summand on the right side is a compact set. By Grothendieck's lemma, so is $T(\{f_n\})$. ∎

## 5. ADDENDUM

This paper was written some well five years ago. In a sense, their results can be seen as folklore. In a sense, they have interest. Several times they have been quoted by other authors and used. It always remained the problem of a proper reference. For this reason I decided to write them out. They appear announced in the Jarandilla Functional Analysis proceedings, and a brief account can be seen in [5].

A result that has been improved deserves mention: thanks to Gonzalo and Jaramillo, and Farmer and Johnson the counterpart of cotype $s \Longrightarrow C_r, \quad r < s^*$ is true: type $s \Longrightarrow W_{s^*}$. However, an interesting question remains open:

$$X \in C_2 \Longleftrightarrow L(\ell_\infty, X) = K(\ell_\infty, X) \ ?$$

Paper [6] contains some remarks about this. The following is a non complete list of reference.

## References


[1] Aron, R., Alencar, R. and Fricke, G., Tensor products of Tsirelson's space, *Illinois J. Math.,* **31** (1) (1987), 17–23.

[2] Bombal, F., Operators on vector sequence spaces, in London Math. Soc. LN **140**, 1989, 94–106.

[3] Bourgain, J., Dunford-Pettis operators on $L_1$ and the Radon-Nikodym property, *Israel J. Math.,* **37** (1-2) (1980), 34–47.

[4] Castillo, J.M.F., $p$-converging and weakly-$p$-compact operators in $L_p$ spaces, in *Extracta Math.,* Vol. especial dedicado al II Congreso de Análisis Funcional, Jarandilla de la Vera, Cáceres, 1990, 46–54.

[5] Castillo, J.M.F. and Sánchez, F., Dunford-Pettis-like properties of continuous function vector spaces, *Rev. Mat. Univ. Complutense Madrid,* **6** (1993), 43–59.

[6] Castillo, J.M.F. and Sánchez, F., Remarks on the range of a vector measure, *Glasgow Math. J.,* **36** (1994), 157–161.

[7] Castillo, J.M.F. and López Molina, J.A., Dunford-Pettis-like properties of projective and natural tensor product spaces, *Rev. Mat. Univ. Complutense Madrid,* **6** (1993), 233–240.

[8] Diestel, J., "Sequences and Series in Banach Spaces", GTM 92, Springer, New York, Heidelberg, Belin, 1984.

[9] Kadec, M.I. and Pelczyǹski, A., Bases, lacunary sequences and complemented subspaces of the spaces $L_p$, *Studia Math.,* **21** (1962), 161–176.

[10] Pietsch, A., "Operator Ideals", North Holland, Amsterdam, 1980.

[11] Rosenthal, H.P., On subspaces of $L_p$, *Ann. of Math.,* **97** (1973), 44–373.

[12] Singer, I., "Bases in Banach Spaces I", Springer, Berlin, 1970.

[13] Weis, L., On perturbations of Fredholm operators in $L_p(\mu)$ spaces, *Proc. Amer. Math. Soc.,* **67** (1977), 287–292.