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Almost Product Structures and Poisson Reduction of Presymplectic Systems *

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1. Introduction

There are two natural generalizations of symplectic manifolds. The first one is obtained by considering the Poisson bracket defined from the symplectic form and gives rise to the notion of Poisson manifold [11, 17]. The second one is obtained by weakening the maximality of the rank and gives rise to the notion of presymplectic manifold [11]. Presymplectic manifolds appear in order to globalize the constrained Dirac-Bergmann formalism for singular Lagrangian systems [3, 5, 6, 7, 2, 1]. In fact, if $L : TQ \longrightarrow \mathbb{R}$ is a singular Lagrangian, then $M_1 = Leg(TQ) \subset T^*Q$ is a submanifold of a symplectic manifold T^*Q, where $Leg : TQ \longrightarrow T^*Q$ is the Legendre transformation. Thus, M_1 is a presymplectic manifold. If L admits a global dynamics there is no secondary constraints and the analysis of the dynamical equations is made on M_1, the submanifold defined by the primary constraints. If there is secondary constraints, Gotay and Nester [5, 6, 7] have developed a constraint algorithm which globalizes the Dirac-Bergmann one and we obtain a final constraint submanifold with a global dynamics. In both cases, we get the same geometrical picture: a presymplectic manifold.

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For quantization it is necessary to have a Poisson bracket on the carrier space of a given dynamical system. However, it is not possible to define in a natural way a Poisson bracket in an arbitrary presymplectic manifold (M, ω). In [4] Dubrovin *et al* have used a so-called generalized connection in the presymplectic manifold in order to define a Poisson bracket by projection. Such a generalized connection is in fact an almost product structure adapted to the presymplectic form ω, i.e., one of the complementary distributions is the characteristic distribution and, the other one is the "regular part" where the dynamics take place. This trick was used by de León and Rodrigues in order to obtain the dynamics for a singular Lagrangian system [8, 9, 10] and independently by Pitanga and Mundin [16, 15] (see also [12, 1]).

The purpose of this Note is to apply the method of almost product structures to perform a Poisson reduction of a presymplectic system. To do this, we assume that there exists an integrable almost structure which is adapted to ω and invariant by the Lie group of symmetries G which acts presymplectically on M. If all the vector fields ξ_M belong to the regular distribution and μ is a regular value of a momentum map $J : M \longrightarrow \mathfrak{g}^*$, then the reduced space $M_\mu = \dfrac{J^{-1}(\mu)}{G_\mu}$ is endowed with a reduced presymplectic form and a reduced almost product structure which defines exactly the reduced Poisson bracket obtained from the induced Poisson bracket on M. In order to introduce the dynamics in this picture, we suppose first that the G-invariant Hamiltonian function H admits a global dynamics, or, in other words, there are no secondary constraints. Thus, the reduced dynamics are obtained. If there are secondary constraints, then we develop a constraint algorithm and obtain a final presymplectic manifold with a global dynamics.

2. Almost product structures adapted to presymplectic structures

Let (M, ω) be a presymplectic manifold of constant rank r, namely ω is a closed 2-form satisfying $\omega^r \neq 0$ and $\omega^{r+1} = 0$. Hence M has dimension $2r + s$, where $s \geq 0$. If $s = 0$, we are in presence of a symplectic manifold.

We say that an almost product structure F is adapted to the presymplectic form ω if

$$\ker \omega = \ker A \ ,$$

where $A = \frac{1}{2}(Id + F)$ and $B = \frac{1}{2}(Id - F)$ are the canonical projectors of F and, $\mathcal{A} = ImA$ and $\mathcal{B} = ImB$ are the complementary distributions defined by

F. We also denote by (A, B) the almost product structure F. Notice that $\mathcal{B}$ is always integrable, since ω is closed. We denote by A^* and B^* the transpose operators and, by $\mathcal{A}^*$ and $\mathcal{B}^*$ their images. In such a case, the restriction of $\flat$ to the distribution $\mathcal{A}$ induces an isomorphism of C^∞-modules:

$$\flat : \mathcal{A} \longrightarrow \mathcal{A}^* .$$

Then, given a one-form α on M, there exists a unique vector field $X_{\alpha,A} \in \mathcal{A}$ such that

$$i_{X_{\alpha,A}}\omega = A^*\alpha . \tag{1}$$

For a function f on M we put $X_{f,A} = X_{df,A}$. Next, we can define a bracket of functions as follows:

$$\{f, g\}_A = -\omega(X_{f,A}, X_{g,A}) ;$$

$\{\, , \,\}_A$ satisfies all the properties of a Poisson bracket except the Jacobi identity. In fact, we have:

LEMMA 2.1.

$$i_{X_{\{f,g\}_A},A}\omega(Z) = i_{[X_{f,A},X_{g,A}]}\omega(Z) + B^*df[X_{g,A}, AZ] - B^*dg[X_{f,A}, AZ] ,$$

$\forall Z \in \mathfrak{X}(M), \forall f, g \in C^\infty(M)$.

PROPOSITION 2.2. *The bracket* $\{\, , \,\}_A$ *defined by the almost product structure* F *satisfies the Jacobi identity if and only if* F *is integrable.*

Proof. If $\{\, , \,\}_A$ satisfies the Jacobi identity then $X_{\{f,g\}_A,A} = [X_{f,A}, X_{g,A}]$, for any two functions f and g on M. Since the vector fields $X_{f,A}$ span $\mathcal{A}$, then $\mathcal{A}$ is integrable. Therefore, the almost product structure F is integrable (see [9]).

Conversely, if F is integrable then, from Lemma 2.1 we deduce that $i_{X_{\{f,g\}_A},A}\omega(Z) = i_{[X_{f,A},X_{g,A}]}\omega(Z)$. Therefore, the vector fields $X_{\{f,g\}_A,A}$ and $[X_{f,A)}, X_{(g,A)}]$ differ by an element of $\ker \omega$. But, since the almost product structure F is integrable, we deduce that $[X_{f,A}, X_{g,A}] \in \mathcal{A}$. Thus, $X_{\{f,g\}_A,A} = [X_{f,A}, X_{g,A}]$. ∎

As a consequence, if we assume that F is integrable, we have a Poisson manifold $(M, \{\, , \,\}_A)$ whose symplectic foliation is just $\mathcal{A}$. Furthermore, the symplectic form on each leaf $\mathcal{L}$ is just the restriction of the presymplectic form to $\mathcal{L}$. If we denote by $\natural_A : T^*M \longrightarrow TM$ the linear mapping defined by $\langle \natural_A(df), dg\rangle = \{f, g\}_A$, then $X_{f,A} = \natural_A(df)$. Thus, $X_{H,A}f = \{H, f\}_A$, for any function f, where $H : M \longrightarrow \mathbb{R}$ is a Hamiltonian function.

Remark 2.3. In [4] an almost product structure was called a generalized connection. The reason for this name is the following. Suppose that $\ker \omega$ is fibrating, i.e., there is well-defined the quotient manifold $\bar{M} = M/\ker \omega$ and we have a fibered manifold $\pi : M \longrightarrow \bar{M}$. Hence $\mathcal{A}$ defines a connection in π in the sense of Ehresmann.

3. Poisson reduction

Definition 3.1. Let $(M, \{\,,\})$ be a Poisson manifold. We say that $\Phi : G \times M \longrightarrow M$ is a Poisson action if Φ preserves the Poisson bracket $\{\,,\}$, i.e.,

$$\{f, h\} \circ \Phi_g = \{f \circ \Phi_g, h \circ \Phi_g\} \ , \forall f, h \in C^\infty(M), \forall g \in G \ .$$

If the action of G is free and proper, then the quotient manifold M/G is a differentiable manifold and $\pi : M \longrightarrow M/G$ is a principal G-bundle, where π denotes the canonical projection. As we know, M/G is a Poisson manifold with Poisson bracket defined as follows:

$$\{\tilde{F}, \tilde{H}\}_{M/G} = \widetilde{\{F, H\}} \ ,$$

for all $\tilde{F}, \tilde{H} \in C^\infty(M/G)$ where we denote by F and H any functions on M projectable onto $\tilde{F}$ and $\tilde{H}$, respectively [14].

A mapping $J : M \longrightarrow \mathfrak{g}^*$ such that any vector field ξ_M is a Hamiltonian vector field for $\widehat{J\xi}$, will be called a momentum map for the Poisson action, where $\mathfrak{g}$ denotes the Lie algebra of G, ξ_M denotes the vector field on M induced by $\xi \in \mathfrak{g}$ and, $\widehat{J\xi}$ is the function defined by $\widehat{J\xi}(x) = \langle J(x), \xi \rangle$. We say that a momentum map is G-equivariant if it verifies that

$$J(\Phi_g(x)) = Ad^*_{g^{-1}} J(x) \ , \forall g \in G, \forall x \in M \ .$$

It is easy to prove that a momentum map for a Poisson action is G-equivariant iff $J : M \longrightarrow \mathfrak{g}^*$ is a Poisson map, where we consider in $\mathfrak{g}^*$ the Lie-Poisson bracket. Also, the G-equivariance for a momentum map of a Poisson action is equivalent to the following condition:

$$\widehat{J[\xi, \nu]} = \{\widehat{J\xi}, \widehat{J\nu}\} \ , \ \forall \xi, \nu \in \mathfrak{g} \ .$$

Suppose now that $\mu \in \mathfrak{g}^*$ is a regular value for J. In that case, $J^{-1}(\mu)$ is a closed regular submanifold of M which is invariant by the isotropy group G_μ. If the action of G_μ is free and proper then $J^{-1}(\mu)$ is a principal G_μ-bundle over the quotient manifold $M_\mu = J^{-1}(\mu)/G_\mu$. We denote by $\pi_\mu : J^{-1}(\mu) \longrightarrow M_\mu$ the canonical projection.

Now, we apply Theorem 6.48 of reference [14].

THEOREM 3.2. *Let M be a Poisson map and let $\Phi : G \times M \longrightarrow M$ be a Poisson action admitting a G-equivariant momentum map $J : M \longrightarrow \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a regular value of J. If G_μ acts freely and properly on the submanifold $J^{-1}(\mu)$ then the reduced space $M_\mu = J^{-1}(\mu)/G_\mu$ has a unique structure of Poisson manifold with Poisson bracket $\{\cdot, \}_\mu$ and there exists an inmersion $\phi : M_\mu \longrightarrow \frac{M}{G}$ such that $\phi(M_\mu)$ is a Poisson submanifold of M/G. Moreover, the following diagram conmutes:*

$$\begin{array}{ccc} J^{-1}(\mu) & \xrightarrow{\ i_\mu\ } & M \\ {\scriptstyle \pi_\mu}\downarrow & & \downarrow{\scriptstyle \pi} \\ M_\mu & \xrightarrow{\ \phi\ } & M/G \end{array}$$

where i_μ is the embedding of $J^{-1}(\mu)$ onto M.

Remark 3.3. If G is connected, then the action of G preserves not only the symplectic foliation, but each of its leaves. Thus, if $\mathcal{L}$ is a symplectic leaf, the restriction of J to $\mathcal{L}$ is a momentum map for the restricted action of G. If μ is still a regular value for the restricted momentum map, we can perform a symplectic reduction of $\mathcal{L}$ with momentum μ. If some conditions of clean intersection are verified, we can relate these symplectic reductions with the Poisson reduction. The Poisson reduction collects, in some sense, all these symplectic reductions (see Vaisman [17] for details).

4. PRESYMPLECTIC REDUCTION

Let (M, ω) be a presymplectic manifold and let G be a Lie group acting presymplectically on M, i.e, $\Phi_g^*\omega = \omega$, $\forall g \in G$, where $\Phi : G \times M \longrightarrow M$ is the action of G.

A mapping $J : M \longrightarrow \mathfrak{g}^*$ such that any vector field ξ_M is a Hamiltonian vector field for $\widehat{J\xi} = \langle J, \xi \rangle$, i.e.,

$$i_{\xi_M}\omega = d(\widehat{J\xi}) \ ,$$

will be called a momentum map for the presymplectic action (see [2]).

PROPOSITION 4.1. *(1) Let $\Phi : G \times M \longrightarrow M$ be a presymplectic action such that it preserves the almost product structure (A, B), that is, $(T\Phi_g)A =$*

$A(T\Phi_g)$, $\forall g \in G$. In that case, Φ is a Poisson action for the Poisson bracket $\{\, , \}_A$.

(2) If $J : M \longrightarrow \mathfrak{g}^*$ is a momentum map for the presymplectic action Φ such that

1. Φ preserves the almost product structure (A, B),

2. $\xi_M \in \mathcal{A}$, $\forall \xi \in \mathfrak{g}$,

then J is a momentum map for the Poisson action $\Phi : G \times M \longrightarrow M$ where, here, we consider M as a Poisson manifold with Poisson bracket $\{\, , \}_A$.

THEOREM 4.2. *Let (M, ω) be a presymplectic manifold endowed with an integrable almost product structure (A, B) adapted to ω and let G be a Lie group acting presymplectically on M. Let $J : M \longrightarrow \mathfrak{g}^*$ be an equivariant momentum mapping for this action. We also suppose that (A, B) is G-invariant and $\xi_M \in \mathcal{A}$, $\forall \xi \in \mathfrak{g}$. Assume that $\mu \in \mathfrak{g}^*$ is a regular value of J and that the isotropy group G_μ acts freely and properly on $J^{-1}(\mu)$. Then, the quotient manifold M_μ has a unique presymplectic form ω_μ such that $\pi_\mu^* \omega_\mu = i_\mu^* \omega$, and a unique almost product structure (A_μ, B_μ) adapted to ω_μ such that the induced Poisson bracket $\{\ ,\ \}_{A_\mu}$ coincides with the Poisson bracket $\{\ ,\ \}_\mu$ obtained from Theorem 3.2.*

Proof. In fact, A and B are tangent to $J^{-1}(\mu)$ and G_μ-invariant. Hence they project onto an almost product structure (A_μ, B_μ) on M_μ. Since the Nijenhuis tensors of A and B project onto the Nijenhuis tensors of A_μ and B_μ, respectively, we deduce that (A_μ, B_μ) is also integrable. Further, the form $i_\mu^* \omega$ is G_μ-invariant and then it projects onto a 2-form ω_μ. That (A_μ, B_μ) is adapted to ω_μ follows from a direct computation. Finally, one can directly check that $\{\, , \}_{A_\mu}$ coincides with $(\{\, , \}_A)_\mu$. ∎

5. REDUCTION OF THE DYNAMICS

Let (M, ω) be a presymplectic manifold and $H : M \longrightarrow \mathbb{R}$ a function. In that case, we say that (M, ω, H) is a presymplectic system with Hamiltonian function H. We seek for a solution of the equation:

$$i_X \omega = dH \ . \tag{2}$$

Since ω is not symplectic, (2) has no solution in general, and even, if it exists, it will be not unique.

In [6], Gotay and Nester have developed a constraint algorithm for presymplectic systems. Consider the points of M where (2) has a solution and suppose that this set M_2 is a submanifold of M. Nevertheless, the solutions of (2) on M_2 are not necessarily tangent to M. Hence, we consider the points of M_2 on which there exists a solution which is tangent to M_2. Thus, we obtain a new submanifold M_3 and the proccess may be continued. We obtain the following sequence of submanifolds:

$$\cdots \to M_k \to \cdots \to M_2 \to M_1 = M \ .$$

Alternatively, these submanifolds can be described as follows:

$$M_i = \{x \in M \mid \forall v \in T_x M_{i-1}^{\perp} \ , \ v(H) = 0\} \ ,$$

where

$$T_x M_{i-1}^{\perp} = \{v \in T_x M \mid \forall u \in T_x M_{i-1} \ , \ \omega(x)(u, v) = 0\} \ .$$

We call M_2 the secondary constraint submanifold, M_3 the tertiary constraint submanifold, and, in general, M_i is the i-ary constraint submanifold.

If the algorithm stabilizes, that is, there exists a positive integer $k \in \mathbb{N}$ such that $M_k = M_{k+1}$ and $\dim M_k \neq 0$, then we have a final submanifold M_f where, by construction, a solution X on M_f exists, i.e., $X \in \mathfrak{X}(M_f)$ verifies that

$$(i_X \omega = dH)_{/M_f} \ . \tag{3}$$

We assume that (M, ω, H) has no secondary constraints. This fact is equivalent to that

$$dH(x)(\ker \omega)(x) = 0 \ , \ \forall x \in M \ .$$

In other words, the presymplectic system admits a global dynamics. In that case, if (A, B) is an adapted almost product structure, we have $A^*(dH) = dH$, and then there exists a unique vector field X on M such that $X \in \mathcal{A}$ and $i_X \omega = dH$.

Next, let G be a Lie group acting presymplectically on M in such a way that the hypotheses of Theorem 4.2. Assume that H is G-invariant. Under these conditions we deduce that X is G-invariant. Since H is G-invariant, then $H \circ i_\mu$ is G_μ-invariant and it projects onto a function H_μ on M_μ. The vector field X is tangent to $J^{-1}(\mu)$ and hence it projects onto a vector field X_μ on M_μ. A direct computation shows that

$$i_{X_\mu} \ \omega_\mu = dH_\mu \ ,$$

Hence, the reduced presymplectic system $(M_\mu, \omega_\mu, H_\mu)$ has no secondary constraints and its dynamics is given by the integral curves of X_μ, i.e., $X_\mu f = \{H_\mu, f\}$, $\forall f \in C^\infty(M_\mu)$.

If (M, ω, H) has secondary constraints, and there exists a final constraint submanifold M_f, we can develop the above procedure for the presymplectic system (M_f, ω_f, H_f), where $\omega_f = \omega_{|M_f}$ and $H_f = H_{|M_f}$ are the restrictions. Notice that the algorithm preserves the action of G and we obtain a presymplectic action of G on M_f.

6. Reconstruction of the dynamics

In order to reconstruct the dynamics from the reduced ones we may use a connection γ in the principal G_μ-bundle $\pi_\mu : J^{-1}(\mu) \longrightarrow M_\mu$ [13]. Let $c_\mu(t)$ be an integral curve of X_μ passing trough a point $x_0 = c_\mu(0)$ and take its horizontal lift $d(t)$ passing trough z_0, say $z_0 = d(0)$, $\pi_\mu(z_0) = x_0$. Therefore, we have

$$\gamma(X(d(t)) - d'(t)) = \gamma(X(d(t))) = \xi(t) ,$$

where $\xi(t)$ is a curve into $\mathfrak{g}_\mu$, the Lie algebra of G_μ. Next, we put $c(t) = g(t)d(t)$, $g(t)$ being a curve in G_μ to be determined in order that $c(t)$ would be an integral curve of X. From the G_μ-invariance of X we deduce that

$$X(d(t)) - d'(t) = [T_{g(t)} L_{g(t)^{-1}}(g'(t))]_{J^{-1}(\mu)}(d(t)) ,$$

which implies that

$$\xi(t) = T_{g(t)} L_{g(t)^{-1}}(g'(t)) ,$$

or, equivalently,

$$g'(t) = T_e L_{g(t)}(\xi(t)) , \tag{4}$$

with $g(0) = e$. Now, we only have to solve (4), which is usually made by quadratures.

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