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Relative Rotundity in $L^p(X)$[†]

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1. Introduction

The concept of uniform rotundity relative to a linear subspace was introduced by H. Fakhoury [5] in order to establish continuity and uniqueness properties of the best approximation projection. In this paper we investigate the lifting of uniform rotundity relative to a linear subspace from a space X to the corresponding Lebesgue–Bochner function spaces. The main result states that, for $1 < p < \infty$, the normed space X is uniformly rotund relative to its linear subspace Y if and only if the Lebesgue–Bochner function space $L^p(X)$ is uniformly rotund relative to $L^p(Y)$. The proof extends to Köthe normed spaces of vector-valued functions and to Day's substitution spaces. The question of when Lebesgue-Bochner function spaces inherit properties of rotundity was began in 1940 by Boas [1] and Day [4], who deal with the uniform rotund case, and was continued in [8, 11, 6, 10], for various generalizations of the uniform rotundity property.

Terminology and notations are standard. Let X be a normed space. As usual, B and S denote the closed unit ball and the unit sphere of X respectively.

The space X is said to be *uniformly rotund relative to the linear subspace* Y *of* X when the relative modulus of rotundity

$$\delta(Y,\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S, x-y \in Y, \|x-y\| \geq \epsilon\right\}$$

is positive for every $0 < \epsilon \leq 2$. Uniform rotundity relative to $Y = X$ is uniform rotundity in Clarkson's sense [2], and uniform rotundity relative to a

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1-dimensional linear subspace $Y = < z >$ is uniform rotundity in the direction z [7, 12].

It is obvious that if X is uniformly rotund relative to Y, then X is uniformly rotund in every direction of Y. Moreover, a compactness argument proves that the two concepts coincide when Y is finite-dimensional. However, the two notions are distinct.

2. Function Spaces

Let (T, Σ, μ) be a measure space, and let $L^p(X)$, $1 \leq p < \infty$ (resp. $p = \infty$), be the Lebesgue–Bochner function space of μ-equivalence classes of strongly measurable functions $x \colon T \to X$ with $\int_T \|x(t)\|^p \, d\mu < \infty$ (resp. μ-essentially bounded), endowed with the norm

$$\|x\| = \left[\int_T \|x(t)\|^p \, d\mu \right]^{1/p}, \quad (\text{resp. } \|x\| = \operatorname*{ess\,sup}_{t \in T} \|x(t)\|).$$

When $X = \mathbb{R}$, we denote $L^p(\mathbb{R}) = L^p$.

The result that follows is an answer to the problem of whether relative uniform rotundity lifts from X to $L^p(X)$.

Theorem 1. *Let Y be a linear subspace of the normed space X. Then, for $1 < p < \infty$, $L^p(X)$ is uniformly rotund relative to $L^p(Y)$ if and only if X is uniformly rotund relative to Y.*

Let E be the Köthe-Banach space of equivalence classes of μ-measurable functions ξ from T into $\mathbb{C}$ such that if $\eta : T \to \mathbb{C}$ is μ-measurable, $\xi \in E$, and $|\eta| \leq |\xi|$, then $\eta \in E$ and $\|\eta\|_E \leq \|\xi\|_E$ (condition (a)). Let X be a normed space. The vector-valued Köthe normed space $E(X)$ is the space of the equivalence classes of Bochner measurable functions x from T into X, such that if $\xi(t) = \|x(t)\|_E$, then $\xi \in E$ (condition (b)). Let T_0 be a μ–measurable subset of T, and set

$$E_0 = \{\xi \in E : \xi = 0 \text{ in } T_0\} .$$

Since the proof of Theorem 1 only uses conditions (a) and (b), the proof carries over for $E(X)$. Thus we have also stated the following result:

Theorem 2. *Let Y be a linear subspace of X. The vector valued Köthe normed space $E(X)$ is uniformly rotund relative to $E_0(Y)$ if and only if E is uniformly rotund relative to E_0 and X is uniformly rotund relative to Y.*

Let (T, Σ, μ) be a σ-finite measure space which contains no atoms, and let either $E = L^1$ or $E = L^\infty$. Since E is no uniformly rotund relative to any E_0 (see [9]), $L^1(X)$ (resp. $L^\infty(X)$) is never uniformly rotund relative to any $L^1_0(Y)$ (resp. $L^\infty_0(Y)$).

3. Sequence Spaces

Let I be an index set. *A full function space* [3, p. 35] is a normed space E of real or complex functions ξ on I such that (condition (a)) for each $\xi \in E$, every function η for which $|\eta(i)| \leq |\xi(i)|$ for all $i \in I$ satisfies $\eta \in E$ and $\|\eta\|_E \leq \|\xi\|_E$. If a normed space $(X_i, \|\cdot\|_i)$ is given for each $i \in I$, let $\mathcal{P}_E X_i$, *the substitution space of the X_i in E*, be the space of all those functions x on I such that (condition (b)) $x_i \in X_i$ for all $i \in I$, and if $\xi(i) = \|x_i\|_i$ for all $i \in I$, then $\xi \in E$. The space $\mathcal{P}_E X_i$ is normed by $\|x\| = \|\xi\|_E$, where $\xi = (\xi_i) = (\|x_i\|_i)$. We shall say that the X_i have a *relative-to-Y_i common modulus of rotundity* if $\inf_{i \in I} \delta(Y_i, \epsilon)$ is strictly positive for every $0 < \epsilon \leq 2$.

In order to include the posibility that Y_i may be the trivial space $\{0\}$, define I_0 be the set of $i \in I$ for which $Y_i = \{0\}$, and let E_0 be the full function subspace of E defined by

$$E_0 = \{(\xi_i) \in E \ : \ \xi_i = 0, \text{ for every } i \in I_0\}.$$

Again, since the proof of Theorem 1 depends only on conditions (a) and (b), it can be restated in the following terms for substitution spaces.

Theorem 3. *The space $\mathcal{P}_E X_i$ is uniformly rotund relative to $\mathcal{P}_{E_0} Y_i$ if and only if E is uniformly rotund relative to E_0 and the X_i have a relative-to-Y_i common modulus of rotundity.*

Let ℓ^p, $1 \leq p < \infty$ (resp. $p = \infty$), be the Banach space of real–valued functions $\xi = (\xi_i)_{i \in I}$ whose pth power is absolutely summable on I, with norm defined by $\|\xi\|_{\ell^p} = \left(\sum_{i \in I} \|\xi_i\|^p\right)^{1/p}$ (resp. which are bounded, with norm $\sup_{i \in I} |\xi_i|$). For every $I_0 \subset I$, set

$$\ell^p_0 = \{\xi \in \ell^p : \xi_i = 0, \text{ for every } i \in I_0\}.$$

If X is a normed space, let us denote $\ell^p(X)$ by $\mathcal{P}_{\ell^p}(X)$ the substitution space formed by setting $E = \ell^p$, and $X_i = X$ for every $i \in I$. Similarly, if Y is a linear subspace of X and $I_0 \subset I$, let us denote $\ell^p_0(Y)$ by $\mathcal{P}_{\ell^p_0}(Y)$.

Then, as a consequence of Theorem 3, we have the following result.

Corollary 4. *Let X be a normed space and Y a linear subspace of X. Then*

(i) *If $1 < p < \infty$, X is uniformly rotund relative to Y if and only if $\ell^p(X)$ is uniformly rotund relative to $\ell^p(Y)$.*

(ii) *If $I_0 = I \setminus \{i\}$ for some $i \in I$, X is uniformly rotund relative to Y if and only if $\ell^1(X)$ is uniformly rotund relative to $\ell_0^1(Y)$. Otherwise $\ell^1(X)$ is no uniformly rotund relative to $\ell_0^1(Y)$.*

(iii) *For every $I_0 \subset I$, $\ell^\infty(X)$ is never uniformly rotund relative to $\ell_0^\infty(Y)$.*

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