Relative Rotundity in $L^p(X)$

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1. Introduction

The concept of uniform rotundity relative to a linear subspace was introduced by H. Fakhouy [5] in order to establish continuity and uniqueness properties of the best approximation projection. In this paper we investigate the lifting of uniform rotundity relative to a linear subspace from a space $X$ to the corresponding Lebesgue–Bochner function spaces. The main result states that, for $1 < p < \infty$, the normed space $X$ is uniformly rotund relative to its linear subspace $Y$ if and only if the Lebesgue–Bochner function space $L^p(X)$ is uniformly rotund relative to $L^p(Y)$. The proof extends to Köthe normed spaces of vector-valued functions and to Day’s substitution spaces. The question of when Lebesgue-Bochner function spaces inherit properties of rotundity was began in 1940 by Boas [1] and Day [4], who deal with the uniform rotund case, and was continued in [8, 11, 6, 10], for various generalizations of the uniform rotundity property.

Terminology and notations are standard. Let $X$ be a normed space. As usual, $B$ and $S$ denote the closed unit ball and the unit sphere of $X$ respectively.

The space $X$ is said to be **uniformly rotund relative to the linear subspace** $Y$ of $X$ when the relative modulus of rotundity

$$\delta(Y, \epsilon) := \inf \left\{ 1 - \frac{x + y}{2} : x, y \in S, x - y \in Y, \|x - y\| \geq \epsilon \right\}$$

is positive for every $0 < \epsilon \leq 2$. Uniform rotundity relative to $Y = X$ is uniform rotundity in Clarkson’s sense [2], and uniform rotundity relative to a

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1-dimensional linear subspace \( Y = \langle z \rangle \) is uniform rotundity in the direction \( z \) [7, 12].

It is obvious that if \( X \) is uniformly rotund relative to \( Y \), then \( X \) is uniformly rotund in every direction of \( Y \). Moreover, a compactness argument proves that the two concepts coincide when \( Y \) is finite-dimensional. However, the two notions are distinct.

2. Function Spaces

Let \((T, \Sigma, \mu)\) be a measure space, and let \( L^p(X) \), \( 1 \leq p < \infty \) (resp. \( p = \infty \)), be the Lebesgue–Bochner function space of \( \mu \)-equivalence classes of strongly measurable functions \( x: T \to X \) with \( \int_T \|x(t)\|^p d\mu < \infty \) (resp. \( \mu \)-essentially bounded), endowed with the norm

\[
\|x\| = \left( \int_T \|x(t)\|^p d\mu \right)^{1/p}, \quad (\text{resp. } \|x\| = \text{ess sup}_{t \in T} \|x(t)\|).
\]

When \( X = \mathbb{R} \), we denote \( L^p(\mathbb{R}) = L^p \).

The result that follows is an answer to the problem of whether relative uniform rotundity lifts from \( X \) to \( L^p(X) \).

**Theorem 1.** Let \( Y \) be a linear subspace of the normed space \( X \). Then, for \( 1 < p < \infty \), \( L^p(X) \) is uniformly rotund relative to \( L^p(Y) \) if and only if \( X \) is uniformly rotund relative to \( Y \).

Let \( E \) be the Köthe-Banach space of equivalence classes of \( \mu \)-measurable functions \( \xi \) from \( T \) into \( \mathbb{C} \) such that if \( \eta: T \to \mathbb{C} \) is \( \mu \)-measurable, \( \xi \in E \), and \( |\eta| \leq |\xi| \), then \( \eta \in E \) and \( \|\eta\|_E \leq \|\xi\|_E \) (condition (a)). Let \( X \) be a normed space. The vector-valued Köthe normed space \( E(X) \) is the space of the equivalence classes of Bochner measurable functions \( x \) from \( T \) into \( X \), such that if \( \xi(t) = \|x(t)\|_E \), then \( \xi \in E \) (condition (b)). Let \( T_0 \) be a \( \mu \)-measurable subset of \( T \), and set

\[
E_0 = \{ \xi \in E : \xi = 0 \text{ in } T_0 \}.
\]

Since the proof of Theorem 1 only uses conditions (a) and (b), the proof carries over for \( E(X) \). Thus we have also stated the following result:

**Theorem 2.** Let \( Y \) be a linear subspace of \( X \). The vector valued Köthe normed space \( E(X) \) is uniformly rotund relative to \( E_0(Y) \) if and only if \( E \) is uniformly rotund relative to \( E_0 \) and \( X \) is uniformly rotund relative to \( Y \).
Let \((T, \Sigma, \mu)\) be a \(\sigma\)-finite measure space which contains no atoms, and let either \(E = L^1\) or \(E = L^\infty\). Since \(E\) is no uniformly rotund relative to any \(E_0\) (see \([9]\), \(L^1(X)\) (resp. \(L^\infty(X)\)) is never uniformly rotund relative to any \(L^1_0(Y)\) (resp. \(L^\infty_0(Y)\)).

3. Sequence Spaces

Let \(I\) be an index set. A full function space \([3, \text{p. 35}]\) is a normed space \(E\) of real or complex functions \(\xi\) on \(I\) such that \((\text{condition (a)})\) for each \(\xi \in E\), every function \(\eta\) for which \(|\eta(i)| \leq |\xi(i)|\) for all \(i \in I\) satisfies \(\eta \in E\) and \(\|\eta\| \leq \|\xi\|\). If a normed space \((X, \| \cdot \|_i)\) is given for each \(i \in I\), let \(P_E X_i\), the substitution space of the \(X_i\) in \(E\), be the space of all those functions \(x\) on \(I\) such that \((\text{condition (b)})\) \(x_i \in X_i\) for all \(i \in I\), and if \(\xi(i) = \|x_i\|_i\) for all \(i \in I\), then \(\xi \in E\). The space \(P_E X_i\) is normed by \(\|x\| = \|\xi\|_E\), where \(\xi = (\xi_i) = (\|x_i\|_i)\). We shall say that the \(X_i\) have a relative-to-\(Y_i\) common modulus of rotundity if \(\inf_{i \in I} \delta(Y_i, \epsilon)\) is strictly positive for every \(0 < \epsilon < 2\).

In order to include the possibility that \(Y_i\) may be the trivial space \(\{0\}\), define \(I_0\) to be the set of \(i \in I\) for which \(Y_i = \{0\}\), and let \(E_0\) be the full function subspace of \(E\) defined by

\[E_0 = \{(\xi_i) \in E : \xi_i = 0, \text{ for every } i \in I_0\}.

Again, since the proof of Theorem 1 depends only on conditions (a) and (b), it can be restated in the following terms for substitution spaces.

**Theorem 3.** The space \(P_E X_i\) is uniformly rotund relative to \(P_{E_0} Y_i\) if and only if \(E\) is uniformly rotund relative to \(E_0\) and the \(X_i\) have a relative-to-\(Y_i\) common modulus of rotundity.

Let \(\ell^p\), \(1 \leq p < \infty\) (resp. \(p = \infty\)), be the Banach space of real-valued functions \(\xi = (\xi_i)_{i \in I}\) whose \(p\)th power is absolutely summable on \(I\), with norm defined by \(\|\xi\|_{\ell^p} = (\sum_{i \in I} |\xi_i|^p)^{1/p}\) (resp. which are bounded, with norm \(\sup_{i \in I} |\xi_i|\)). For every \(I_0 \subset I\), set

\[\ell^p_{0} = \{\xi \in \ell^p : \xi_i = 0, \text{ for every } i \in I_0\}.

If \(X\) is a normed space, let us denote \(\ell^p(X)\) by \(P_{\ell^p}(X)\) the substitution space formed by setting \(E = \ell^p\), and \(X_i = X\) for every \(i \in I\). Similarly, if \(Y\) is a linear subspace of \(X\) and \(I_0 \subset I\), let us denote \(\ell^p_0(Y)\) by \(P_{\ell^p_0}(Y)\).

Then, as a consequence of Theorem 3, we have the following result.
COROLLARY 4. Let $X$ be a normed space and $Y$ a linear subspace of $X$. Then

(i) If $1 < p < \infty$, $X$ is uniformly rotund relative to $Y$ if and only if $\ell^p(X)$ is uniformly rotund relative to $\ell^p(Y)$.

(ii) If $I_0 = I \setminus \{i\}$ for some $i \in I$, $X$ is uniformly rotund relative to $Y$ if and only if $\ell^1(X)$ is uniformly rotund relative to $\ell^1_0(Y)$. Otherwise $\ell^1(X)$ is no uniformly rotund relative to $\ell^1_0(Y)$.

(iii) For every $I_0 \subset I$, $\ell^\infty(X)$ is never uniformly rotund relative to $\ell^\infty_0(Y)$.

REFERENCES