EXTRACTA MATHEMATICAE Vol. **10**, Núm. 1, 68 – 71 (1995)

Some Remarks on Sufficiency, Invariance and Conditional Independence*

A.G. Nogales and J.A. Oyola

Dpto. de Matemáticas, Universidad de Extremadura, 06071-*Badajoz, Spain*

(Presented by M. Molina)

AMS *Subject Class.* (1991): 62B05, 62A05 *Received July* 1, 1995

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a statistical experiment (i.e., $\mathcal{P}$ is a family of probability measures on the measurable space $(\Omega, \mathcal{A})$) and G a group of bijective and bimeasurable maps of $(\Omega, \mathcal{A})$ onto itself leaving the family $\mathcal{P}$ invariant, that is, $gP \in \mathcal{P}, \forall P \in \mathcal{P}, \forall g \in \mathcal{G}$, where gP is the probability measure on $\mathcal{A}$ defined by $gP(A) = P(g^{-1}A)$, $A \in \mathcal{A}$. If $P \in \mathcal{P}$, two events $B, C \in \mathcal{A}$ are said to be P–equivalent (and we shall write $B \overset{P}{\sim} C$) if $P(B \triangle C) = 0$; these events are said to be equivalent (we write $B \sim C$) if they are P–equivalent for all $P \in \mathcal{P}$. Let $\mathcal{A}_I = \{A \in \mathcal{A} : gA = A, \forall g \in \mathcal{G}\}$ be the σ–field of G–invariant sets and $\mathcal{A}_A = \{A \in \mathcal{A} : gA \sim A, \forall g \in G\}$ the σ–field of $\mathcal{P}$–almost–G–invariant sets. $\mathcal{A}_S$ will always be a sufficient sub–σ–field of $\mathcal{A}$. Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be three sub–σ–fields of $\mathcal{A}$; for $P \in \mathcal{P}$, the σ–fields $\mathcal{B}$ and $\mathcal{C}$ are said to be P–conditionally independents given $\mathcal{D}$, and we shall write $\mathcal{B} \perp\!\!\!\perp_P \mathcal{C} \,|\mathcal{D}$, if

$$E_P(I_{B\cap C}|\mathcal{D}) \overset{P}{\sim} E_P(I_B|\mathcal{D}) \cdot E_P(I_C|\mathcal{D})$$

for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$. It is well known that $\mathcal{B} \perp\!\!\!\perp_P \mathcal{C} \,|\mathcal{D}$ if and only if

$$E_P(I_C|\mathcal{B} \vee \mathcal{D}) \overset{P}{\sim} E_P(I_C|\mathcal{D}), \forall C \in \mathcal{C},$$

where $\mathcal{B} \vee \mathcal{D}$ is the smallest σ–field containing $\mathcal{B}$ and $\mathcal{D}$. The σ–fields $\mathcal{B}$ and $\mathcal{C}$ are said to be conditionally independents given $\mathcal{D}$, and we shall write $\mathcal{B} \perp\!\!\!\perp \mathcal{C} \,|\mathcal{D}$, if $\mathcal{B} \perp\!\!\!\perp_P \mathcal{C} \,|\mathcal{D}, \forall P \in \mathcal{P}$. Other concepts not defined here can be found, for example, in Lehmann (1986).

This paper is concerned with propositions i) — iv) below.

*A full version of this paper is to appear in The Annals of Statistics.

i) For every $A \in \mathcal{A}_I$, there exists an $\mathcal{A}_S \cap \mathcal{A}_I$–measurable function P–equivalent to $E(I_A|\mathcal{A}_S)$ for every $P \in \mathcal{P}$.

ii) $\mathcal{A}_S \perp\!\!\!\perp \mathcal{A}_I \,|\mathcal{A}_S \cap \mathcal{A}_I$.

iii) $\mathcal{A}_S \cap \mathcal{A}_I$ is sufficient for $\mathcal{P}_{|\mathcal{A}_I}$.

iv) For every $A \in \mathcal{A}_I$, $E(I_A|\mathcal{A}_S)$ is almost invariant.

These propositions have been considered in relation with a theorem of C. Stein on sufficiency and invariance. In Hall, Wijsman and Ghosh (1965) Stein's theorem is stated as follows: "Under conditions A.i) $g\mathcal{A}_S = \mathcal{A}_S, \forall g \in G$ and A.ii) $\mathcal{A}_S \cap \mathcal{A}_I \sim \mathcal{A}_S \cap \mathcal{A}_A$, we have that $\mathcal{A}_S \cap \mathcal{A}_I$ is sufficient for $\mathcal{P}_{|\mathcal{A}_I}$". They pose the question if condition A.ii) can be droped in this result. Example 1 of Landers and Rogge (1973) solve this problem in the negative. This example will be frequently referred to in this paper, where results and counterexamples are given to clarify the relationship between propositions above.

The next theorem states positive results between them.

THEOREM.
a) i) $\Longleftrightarrow$ ii) + iii).
b) ii) $\Longrightarrow$ iv).

REMARKS 1. 1) Lemma 3.3 of Hall, Wijsman and Ghosh (1965) states that i) and ii) are equivalent, but only the implication i) $\Longrightarrow$ ii) is true: Example 1 of Landers and Rogge (1973) shows that ii) $\not\Longrightarrow$ i). Nevertheless, proposition i) is implied by the stronger condition that $\mathcal{A}_S \perp\!\!\!\perp_Q \mathcal{A}_I \,|\mathcal{A}_S \cap \mathcal{A}_I$ for a privileged dominating probability Q on $\mathcal{A}$.

2) The reader can also find in Hall, Wijsman and Ghosh (1965, pp. 595,602) the erroneous assertion that ii) $\Longrightarrow$ iii): Example 1 of Landers and Rogge (1973) is a counterexample. See also Remark 2.1 below.

3) An equivalent formulation of proposition i) is that $E(I_A|\mathcal{A}_S)$ has an invariant version if $A \in \mathcal{A}_I$. To obtain Stein's theorem, Hall, Wijsman and Ghosh (1965, Lemma 3.2) show that A.i)+A.ii) $\Longrightarrow$ i); their proof is also valid to prove the strictly stronger statement that $E(I_A|\mathcal{A}_S)$ has an invariant version for all $A \in \mathcal{A}_A$. Last statement is, in fact, equivalent to A.ii) in presence of A.i).

4) Lemma 3.1 of Hall, Wijsman and Ghosh (1965) shows that proposition iv) is true under the hypothesis $g\mathcal{A}_S = \mathcal{A}_S, \forall g \in G$.

5) The proof of the equivalence of propositions (i) and (ii) in Lemma 3 of Berk (1972) sends the reader to Lemma 3.3 of Hall, Wijsman and Ghosh

(1965), which is false as it is pointed out in Remark 1 above . Nevertheless, an analogous argument to that used in the proof of the part b) of the Theorem also shows that Lemma 3 of Berk (1972) remains true.

The next example shows that neither iii) $\Longrightarrow$ ii) nor iv) $\Longrightarrow$ ii).

EXAMPLE 1. Let us consider the statistical experiment

$$(\mathbb{R}^3, \mathcal{R}^3, \{P_\mu/\mu \in \mathbb{R}\}).$$

where $\mathcal{R}^3$ is the Borel σ–field on $\mathbb{R}^3$ and P_μ is the trivariate normal distribution with mean $(\mu, 2\mu, 0)$ and covariance matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

If X, Y and Z denote the coordinates on $\mathbb{R}^3$, the σ–field $\mathcal{A}_S$ induced by the statistics (X, Y) is sufficient for this experiment. Let G be the group of all bijective maps of $\mathbb{R}^3$ onto itself moving at most a finite set of $\mathbb{R}^3$ and leaving (Y, Z) invariant. The σ–field $\mathcal{A}_S \cap \mathcal{A}_I = Y^{-1}(\mathcal{R})$ is sufficient for $\mathcal{P}_{|\mathcal{A}_I}$. Furthermore, proposition iv) is satisfied because $\mathcal{A}_A = \mathcal{R}^3$. Nevertheless, $\mathcal{A}_S$ and $\mathcal{A}_I$ are not conditionally independent given $\mathcal{A}_S \cap \mathcal{A}_I$ since Z is invariant and $E(Z|\mathcal{A}_I) = 0$ is not equivalent to $E(Z|\mathcal{A}_S \cap \mathcal{A}_I) = (2X - Y)/3$. ■

REMARKS 2. 1) In this example, condition A.i) is not satisfied; but $g\mathcal{A}_S \sim \mathcal{A}_S, \forall g \in G$. Nevertheless, the implication A.i)+iii)+iv) $\Longrightarrow$ ii) is not true either: replacing $\mathcal{A}_S$ by $\mathcal{A}_S \vee \mathcal{N}$ ($\mathcal{N}$ being the family of null Borel sets), a counterexample is obtained. Note that the sufficient statistics (X, Y) is not complete; in fact, it is pointed out in Hall, Wijsman and Ghosh (1965, p. 602) that iii) + completeness of $\mathcal{A}_S$ implies ii).

2) The above mentioned example of Landers and Rogge (1973) shows that iii) is not implied by iv).

The next example shows that implication iii) $\Longrightarrow$ iv) is also false.

EXAMPLE 2. Let $\Omega = \{-2, -1, 1, 2\}$, $\mathcal{A}$ be the family of all subsets of Ω and $\mathcal{P} = \{\epsilon_+, \epsilon_-\}$, where $\epsilon_+ = \frac{1}{2}(\epsilon_1 + \epsilon_2)$ and $\epsilon_- = \frac{1}{2}(\epsilon_{-1} + \epsilon_{-2})$, ϵ_i being the probability measure concentrated at point i. The smallest σ–field $\mathcal{A}_S$ containing the subsets $\{-1\}$ and $\{-2\}$ is sufficient for this experiment. The family $\mathcal{P}$ remains invariant under the action of the group $G = \{I, Z\}$ where I is the identity map on Ω and $Zi = -i$, $i \in \Omega$, and $\mathcal{A}_A = \mathcal{A}_I$ is the smallest σ–field containing $\{-1, 1\}$. The σ–field $\mathcal{A}_S \cap \mathcal{A}_I$ is sufficient for

$\mathcal{A}_I$ since the restrictions of ϵ_+ and ϵ_- to $\mathcal{A}_I$ coincide. On the other hand, the event $\{-1, 1, 2\}$ is not almost–invariant and its indicator function is a common version of $\epsilon_\pm(\{-1,1\}|\mathcal{A}_S)$. ■

Acknowledgments

We wish to thank an Associate Editor, the referee and Profs. J.K. Ghosh and R.A. Wijsman for their helpful comments on an earlier draft of the paper.

References

[1] Berk, R.H., A note on sufficiency and invariance, *Ann. Math. Statist.,* **43** (2) (1972), 647–650.
[2] Hall, W.J., Wijsman, R.A. and Ghosh, J.K., The relationship between sufficiency and invariance, *Ann. Math. Statist.,* **36** (1965), 575–611.
[3] Landers, D. and Rogge, L., On sufficiency and invariance, *Ann. Statist.,* **1** (3) (1973), 543–544.
[4] Lehmann, E.L., "Testing Statistical Hypotheses", 2nd ed., Wiley, New York, 1986.