Some Remarks on Hadamard’s Inequalities for Convex Functions

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1. Introduction

In paper [4] we introduced the following two mappings associated to a convex function $f : [a, b] \to \mathbb{R}$; $H, F : [0, 1] \to \mathbb{R}$ given by

\[
H(t) := \frac{1}{b-a} \int_a^b f \left( tx + (1-t)\frac{a+b}{2} \right) \, dx
\]

and

\[
F(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f \left( tx + (1-t)y \right) \, dx \, dy
\]

and we proved the following main properties

(i) $H, F$ are convex in $[0, 1]$.

(ii) $H$ increases monotonically on $[0, 1]$, $F$ is decreasing on $[0, 1/2]$ and increasing on $[1/2, 1]$.

(iii) We have the bounds

\[
\inf_{t \in [0,1]} H(t) = H(0) = f \left( \frac{a+b}{2} \right) ;
\]

\[
\sup_{t \in [0,1]} H(t) = H(1) = \frac{1}{b-a} \int_a^b f(x) \, dx ;
\]

\[
\sup_{t \in [0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a} \int_a^b f(x) \, dx ;
\]

\[
\inf_{t \in [0,1]} F(t) = F(1/2) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f \left( \frac{x+y}{2} \right) \, dx \, dy .
\]

(iv) One has the inequalities

\[
f \left( \frac{a+b}{2} \right) \leq F(1/2) \quad \text{and} \quad H(t) \leq F(t) \quad \text{for all } t \in [0, 1].
\]
The main aim of this note is to give another type of refinements to the classical inequality due to Hadamard

\[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \]

For other inequalities connected with this main result in Mathematical Analysis, we send to the recent papers [1–10] where further references are given.

2. The Main Results

Let \([a, b]\) be a compact interval of real numbers, \(d := \{x_i \mid i = 0, n\} \subset [a, b]\) a division of the interval \([a, b]\) given by

\[ d : a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \quad (n \geq 1) \]

and \(f\) a bounded mapping on \([a, b]\). We consider the following sums

\[ h_d(f) := \sum_{i=0}^{n-1} f \left( \frac{x_i + x_{i+1}}{2} \right) (x_{i+1} - x_i) \quad \text{(called Hadamard's inferior sum)} \]

\[ H_d(f) := \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \quad \text{(called Hadamard's superior sum)} \]

and Darboux's sums

\[ s_d(f) := \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i) \quad S_d(f) := \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i) \]

where

\[ m_i = \inf_{x \in [x_i, x_{i+1}]} f(x) \quad M_i = \inf_{x \in [x_i, x_{i+1}]} f(x) \quad i = 0, \ldots, n - 1. \]

It is well-known that \(f\) is Riemann integrable on \([a, b]\) if and only if

\[ \sup_d s_d(f) = \inf_d S_d(f) = I \in \mathbb{R} \]

and in this case

\[ I = \int_a^b f(x) \, dx. \]

The following theorem holds:

**Theorem.** Let \(f : [a, b] \to \mathbb{R}\) be a convex function on \([a, b]\). Then

(i) \(h_d(f)\) increases monotonically over \(d\), i.e. for \(d_1 \subseteq d_2\) one has \(h_{d_1}(f) \leq h_{d_2}(f)\).

(ii) \(H_d(f)\) is decreasing over \(d\).
(iii) We have the bounds

\[
(1) \quad \frac{1}{b - a} \inf_x h_d(f) = f\left(\frac{a + b}{2}\right), \quad \sup_x h_d(f) = \int_a^b f(x) \, dx
\]

and

\[
(2) \quad \inf_x H_d(f) = \int_a^b f(x) \, dx, \quad \sup_x H_d(f) = \frac{f(a) + f(b)}{2}.
\]

**Proof.** (i) Without lost of generality we can assume that \(d_1 \subseteq d_2\) with \(d_1 = \{x_0, \ldots, x_n\}\) and \(d_2 = \{x_0, \ldots, x_k, y, x_{k+1}, \ldots, x_n\}\) where \(y \in [x_k, x_{k+1}]\) \((0 \leq k \leq n - 1)\). Then

\[
h_{d_2}(f) - h_{d_1}(f) = f\left(\frac{x_k + y}{2}\right)(y - x_k) + f\left(\frac{y + x_{k+1}}{2}\right)(x_{k+1} - y) - f\left(\frac{x_k + x_{k+1}}{2}\right)(x_{k+1} - x_k).
\]

Let put

\[
\alpha = \frac{y - x_k}{x_{k+1} - x_k}, \quad \beta = \frac{x_{k+1} - y}{x_{k+1} - x_k}, \quad x = \frac{x_k + y}{2}, \quad z = \frac{y + x_{k+1}}{2}.
\]

Then

\[
\alpha + \beta = 1, \quad \alpha x + \beta y = \frac{x_k + x_{k+1}}{2}
\]

and by the convexity of \(f\) we deduce that \(\alpha f(x) + \beta f(z) \geq f(\alpha x + \beta z)\), i.e. \(h_{d_2}(f) \geq h_{d_1}(f)\).

(ii) For \(d_1, d_2\) as above, we have

\[
H_{d_2}(f) - H_{d_1}(f) = \frac{f(x_k) + f(y)}{2}(y - x_k) + \frac{f(y) + f(x_{k+1})}{2}(x_{k+1} - y) + \frac{f(x_k) + f(x_{k+1})}{2}(x_{k+1} - x_k) - \frac{f(y)(x_{k+1} - x_k) + f(x_k)(x_{k+1} - y) + f(x_{k+1})(y - x_k)}{2}.
\]

Now, let \(\alpha, \beta\) be as above and \(u = x_k, v = x_{k+1}\). Then \(\alpha u + \beta v = y\) and by the convexity of \(f\) we have \(\alpha f(u) + \beta f(v) \geq f(y)\), i.e. \(H_{d_2}(f) \leq H_{d_1}(f)\) and the statement is proved.

(iii) Let \(d = \{x_0, \ldots, x_n\}\) with \(a = x_0 < x_1 < \cdots < x_n = b\). Put \(p_i := x_{i+1} - x_i, u_i := (x_{i+1} + x_i)/2, i = 0, \ldots, n - 1\). Then by Jensen’s discrete inequality

\[
f\left(\sum_{i=0}^{n-1} \frac{p_i u_i}{p_i}\right) \leq \frac{\sum_{i=0}^{n-1} p_i f(u_i)}{\sum_{i=0}^{n-1} p_i}.
\]
and since
\[ \sum_{i=0}^{n-1} p_i = b - a, \quad \sum_{i=0}^{n-1} p_i u_i = \frac{b^2 - a^2}{2}, \]
we deduce the inequality
\[ f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} h_d(f). \]

If \( d = d_0 = \{a, b\} \), we obtain \( h_{d_0}(f) = f\left((a + b)/2\right) \), which proves the first bound in (1).

By the first inequality in Hadamard’s result, we have
\[ f\left(\frac{x_i + x_{i+1}}{2}\right) \leq \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x) \, dx, \quad i = 0, \ldots, n - 1, \]
which gives, by addition,
\[
\begin{aligned}
    h_d(f) &= \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i) \\
    &\leq \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) \, dx \\
    &= \int_a^b f(x) \, dx,
\end{aligned}
\]
for all \( d \) a division of \([a, b]\).

Since
\[ s_d(f) \leq h_d(f) \leq \int_a^b f(x) \, dx, \quad d \text{ is a division of } [a, b], \]
and \( f \) is Riemann integrable on \([a, b]\), i.e.
\[ \sup_d s_d(f) = \int_a^b f(x) \, dx, \]

it follows that
\[ \sup_d h_d(f) = \int_a^b f(x) \, dx, \]
which proves the relation (1).

To prove the relation (2), we observe, by the second inequality in Hadamard’s result, that
\[
\begin{aligned}
    \int_a^b f(x) \, dx &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) \, dx \\
    &\leq \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i) \\
    &= H_d(f)
\end{aligned}
\]
where \( d \) is an arbitrary division of \([a, b]\).

Since
\[
H_d(f) \leq S_d(f), \quad \text{for all } d \text{ as above},
\]
and \( f \) is integrable on \([a, b]\), we conclude that
\[
\inf_d H_d(f) = \int_a^b f(x) \, dx
\]
Finally, because for all \( d \) a division of \([a, b]\) we have \( d \supseteq d_0 = \{a, b\} \), thus
\[
\sup_d H_d(f) = \frac{f(a) + f(b)}{2}
\]
and the theorem is proved. \( \blacksquare \)

Remark. Let \( f \) be a convex mapping on \([a, b]\). Then for all \( a = x_0 < x_1 < \cdots < x_n = b \), we have the following improvement of Hadamard’s result

\[
(3) \quad f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) (x_{i+1} - x_i)
\]

\[
\leq \frac{1}{b - a} \int_a^b f(x) \, dx
\]

\[
\leq \frac{1}{b - a} \sum_{i=0}^{n-1} f(x_i) + f(x_{i+1}) \frac{(x_{i+1} - x_i)}{2}
\]

\[
\leq \frac{f(a) + f(b)}{2}.
\]

Corollary 1. Let \( f \) be as above. Define the sequences

\[
H_n(f) := \frac{1}{2n} \sum_{i=0}^{n-1} \left[ f\left(a + \frac{i}{n}(b - a)\right) + f\left(a + \frac{i+1}{n}(b - a)\right)\right]
\]

for \( n \geq 1 \). Then we have the inequalities

\[
(4) \quad f\left(\frac{a + b}{2}\right) \leq h_n(f) \leq \frac{1}{b - a} \int_a^b f(x) \, dx
\]

\[
\leq H_n(f) \leq \frac{f(a) + f(b)}{2}, \quad n \geq 1.
\]

Moreover, one has

\[
(5) \quad \lim_{n \to \infty} h_n(f) = \lim_{n \to \infty} H_n(f) = \frac{1}{b - a} \int_a^b f(x) \, dx.
\]
Proof. The inequalities (4) follows by (3) for \( d := \{ x_i = a + \frac{i}{2^n} (b - a) \mid i = 0, n \} \). The relation (5) is obvious by the integrability of \( f \). We omit the details. \( \square \)

Corollary 2. Let \( f : [a, b] \to \mathbb{R} \) be a convex mapping on \([a, b]\). Define the sequences

\[
 t_n(f) := \frac{1}{2^n} \sum_{i=0}^{n-1} f \left( a + \frac{2^i}{2^{n+1}} (b - a) \right) 2^i
\]

and

\[
 T_n(f) := \frac{1}{2^{n+1}} \sum_{i=0}^{n-1} \left[ f \left( a + \frac{2^i}{2^n} (b - a) \right) + f \left( a + \frac{2^{i+1}}{2^n} (b - a) \right) \right] 2^i
\]

\((n \geq 1)\). Then we have

(i) \( t_n \) is monotonously increasing;

(ii) \( T_n \) is monotonously decreasing;

(iii) The following identities are valid

\[
 \sup_{n \geq 1} t_n(f) = \lim_{n \to \infty} t_n(f) = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

\[
 \inf_{n \geq 1} T_n(f) = \lim_{n \to \infty} T_n(f) = \frac{1}{b-a} \int_a^b f(x) \, dx .
\]

Proof. (i), (ii). Is obvious by (i) and (ii) of Theorem for

\[
 d_n := \left\{ x_i = a + \frac{2^i}{2^n} (b - a) \mid i = 0, n \right\} \subseteq d_{n+1}, \quad n \in \mathbb{N}.
\]

(iii). It follows from bounds (1), (2) and the fact that \( f \) is Riemann integrable on \([a, b]\).

Applications. a) Let \( 0 \leq a = x_0 < x_1 < \cdots < x_n = b \) and \( p \geq 1 \). Then we have the inequalities

\[
 \left( \frac{a + b}{2} \right)^p \leq \frac{1}{b-a} \sum_{i=0}^{n-1} \left( \frac{x_{i+1} + x_i}{2} \right)^p (x_{i+1} - x_i) \leq \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}
\]

\[
 \leq \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{x_i^p + x_{i+1}^p}{2} (x_{i+1} - x_i) \leq \frac{a^p + b^p}{2}.
\]

b) Suppose that \( 0 < a \) and \( x_i \) are as above. Then one has

\[
 \frac{2}{a+b} \leq \frac{2}{b-a} \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{x_{i+1} + x_i} \leq \frac{\ln b - \ln a}{b-a}
\]

\[
 \leq \frac{1}{b-a} \sum_{i=0}^{n-1} \frac{x_{i+1}^2 - x_i^2}{2x_{i+1}} \leq \frac{a+b}{2ab}.
\]
c) We have the following refinement of arithmetic mean-geometric mean inequality

\[
\frac{a + b}{2} \geq \prod_{i=0}^{n-1} \left( \frac{x_i + x_{i+1}}{2} \right)^{\frac{(x_{i+1} - x_i)}{b-a}} \geq \frac{1}{c} \left( \frac{b^a}{a^b} \right)^{\frac{1}{b-a}} \\
\geq \prod_{i=0}^{n-1} \left( \frac{x_i x_{i+1}}{2(b-a)} \right) \geq \sqrt{ab}
\]

where \( a > 0 \) and \( x_i \) are as above.

References