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An Application of the Kronecker Limit Formula

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1. INTRODUCTION AND PRELIMINARY RESULTS

Let $d \neq 1$ be a square free positive rational integer, such that the corresponding imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ is of class–number one. Set $K := \mathbb{Q}(\sqrt{-d})$.

Corresponding to $\mathbb{Q}$ is the well–known Euler constant γ given by:

$$\gamma = \lim_{n \to +\infty} \left(\sum_{p=1}^{n} \frac{1}{p} - \operatorname{Log} n \right).$$

Our aim here is to use the Kronecker first formula (see [7] §1) and a result from the work of Chowla and Selberg [2] to get the explicit value of γ_K, i.e., the analogue of the Euler constant corresponding to the field K.

DEFINITION. The Dedekind zeta–function is defined for $\operatorname{Re}(s) > 1$ by:

$$\zeta_K(s) = \sum_{\alpha} \frac{1}{(N\alpha)^s},$$

where α runs through all integral divisors of the field K and $N\alpha$ denotes the norm of the divisor α.

PROPOSITION. (See [3] §8 or [7] §1) *The Dedekind zeta–function $\zeta_K(s)$ has a meromorphic continuation to the whole s–plane, with a simple pole at $s = 1$. Furthermore we have:*

$$\operatorname{Res}(\zeta_K(s), s=1) := \rho_K = 2\pi \Big/ \omega_K \sqrt{d},$$

where ω_K denotes the number of roots of 1 contained in K.

Remark. To compute ρ_K we have used the following facts:

i) The general formula for the residue given in [3].

ii) The regulator for imaginary quadratic fields is, by definition, equal to 1.

iii) If r_1 and $2r_2$ denote, in general and respectively, the real and complex embeddings of a number field in $\mathbb{C}$, then for our particular field K it is, clearly, $r_1 = 0$ and $r_2 = 1$.

DEFINITION. The Euler constant γ_K corresponding to the imaginary quadratic field K is given by:

$$\gamma_K = \lim_{n\to+\infty} \left(\sum_{N\alpha<n} \frac{1}{N\alpha} - \pi \operatorname{Log} n \right),$$

where the sum is taken over the integral ideals α of K whose norm $N\alpha$ is smaller than the integer n.

PROPOSITION. ([7]) *For s near 1, the meromorphic function $\zeta_K(s)$ admits the following Laurent series*:

$$\zeta_K(s) = \frac{\rho_K}{s-1} + \frac{1}{\omega_K}\gamma_K + o(s-1),$$

where γ_K is the Euler constant corresponding to the field K.

Now, from [4] we get, for $\mathrm{Re}(s) > 1/2$:

$$\zeta_K(s) = \frac{1}{\omega_K} \left[\frac{2}{\sqrt{d}} \right]^s E(\tau, s), \qquad (*)$$

where $E(\tau,s)$ is the Eisenstein series for the following positive definite binary quadratic form:

$$Q(u,v) = y^{-1}(u^2|\tau|^2 + 2uv\mathrm{Re}(\tau) + v^2)$$

in which the complex number $\tau := x + iy$ is given by the formula ([6]):

$$\tau = \begin{cases} i\sqrt{d} & \text{if } d \equiv 1 \text{ or } 2 \pmod 4 \\ \frac{1+i\sqrt{d}}{2} & \text{if } d \equiv 3 \pmod 4 \end{cases}.$$

Hence, we get:

$$\zeta_K(s) = \frac{1}{\omega_K} \left[\frac{2}{\sqrt{d}} \right]^s \sum\nolimits'_{(m,n)\in\mathbb{Z}^2} \frac{y^s}{|m+n\tau|^{2s}},$$

where the dash indicates that $m = n = 0$ is excluded from the summation.

2. MAIN RESULT

We prove the following:

THEOREM. *Let $d \neq 1$ be a square free integer and let $K := \mathbb{Q}(\sqrt{-d})$ be the* *corresponding imaginary quadratic field whose class–number is assumed to be* 1.

Then, the Euler constant γ_K *corresponding to* K *is given by*:

$$\gamma_K = \frac{4\pi}{\sqrt{d}}\left[\gamma + \frac{1}{2}\operatorname{Log}\sqrt{\frac{\pi\sqrt{d}}{y}} - \frac{\omega}{4}\sum_{m=1}^{d}\left(\frac{d}{m}\right)\operatorname{Log}\Gamma\left(\frac{m}{d}\right)\right],$$

where $\left(\frac{d}{m}\right)$ *is the Kronecker symbol and* ω *is a constant depending on* K *and given in the proof below.*

Proof. By the Kronecker first limit formula we have:

$$\lim_{s\to 1}\left[E(s,\tau) - \frac{\pi}{s-1}\right] = 2\pi\left(\gamma - \operatorname{Log}2 - \operatorname{Log}(\sqrt{y}\,|\eta(\tau)|^2)\right),$$

where γ is the usual Euler constant (i.e., corresponding to $\mathbb{Q}$), $y := \operatorname{Im}\tau > 0$, and $\eta(z)$ is the Dedekind eta–function defined for $\operatorname{Im}(z) > 0$ by:

$$\eta(z) = \exp\left[\frac{i\pi z}{12}\right]\prod_{i=0}^{\infty}\left(1 - e^{2\pi i z}\right) \qquad (\operatorname{Im}(z) > 0).$$

From the known relation between the Dedekind zeta–function and the Eisenstein series, we deduce:

$$\gamma_K = \frac{4\pi}{\sqrt{d}}\left(\gamma - \frac{1}{2}\operatorname{Log}(2\sqrt{d}) - \operatorname{Log}(\sqrt{y}\,|\eta(\tau)|^2)\right),$$

whence:

$$\gamma_K = \frac{4\pi}{\sqrt{d}}\left(\gamma - \frac{1}{2}\operatorname{Log}(2\sqrt{d}) - \operatorname{Log}(\sqrt{y}) - \operatorname{Log}(2\pi|\Delta(\tau)|^{1/12})\right),$$

where, as in the theories of modular and elliptic functions, the discriminant $\Delta(z)$ is defined by:

$$\Delta(z) = (2\pi)^{12}(\eta(z))^{24}.$$

Moreover, the Chowla–Selberg formula ([2], p. 110) gives:

$$\Delta(\tau) = \frac{1}{(2\pi)^{18}d^6}\left\{\prod_{m=1}^{d}\Gamma\left(\frac{m}{d}\right)^{\left(\frac{d}{m}\right)}\right\}^{3\omega},$$

where $\left(\frac{d}{m}\right)$ is the Kronecker symbol ([5], p. 89) and:

$$\omega = \begin{cases} 6 \text{ if } d = 3 \\ 4 \text{ if } d = 4 \\ 2 \text{ otherwise}. \end{cases}$$

Combining the final results obtained for γ_K and $\Delta(\tau)$, we easily deduce the announced result for the Euler constant. ∎

3. Explicit Computations for the Cyclotomic Field $\mathbb{Q}(j)$

In [1], and through the study of some elliptic integrals, we obtained modular identities closely related to the lattice $\mathbb{Z} + j\mathbb{Z}$ where j denotes the primitive cubic root of unity. Among these identities we got the explicit value of the Weierstrass invariant $g_3(1,j)$ ([1], p. 420; it is well-known that $g_2(1,j) = 0$).

As an application of this, we get the following:

PROPOSITION. *The Euler constant corresponding to the cyclotomic field $\mathbb{Q}(j)$ is*:

$$\gamma_{\mathbf{Q}(j)} = \frac{4\pi}{\sqrt{3}} \left(\gamma + \frac{1}{4} \operatorname{Log} 3 + 2 \operatorname{Log} \pi - \Gamma\left(\tfrac{1}{3}\right)^3 \right).$$

Proof. Using the result of the theorem above and results about cyclotomic fields ([8] §11), we easily get:

$$\gamma_{\mathbf{Q}(j)} = \frac{4\pi}{\sqrt{3}} \left(\gamma - \operatorname{Log} 4 + \operatorname{Log} \sqrt{3} - \operatorname{Log} |\eta(j)|^2 \right).$$

Moreover, the discriminant $\Delta(z)$ is related to the Weierstrass invariants by the formula:

$$\Delta(z) = g_2^3(z) - 27 g_3^2(z) ,$$

and since:

$$g_3(1,j) = \frac{1}{(2\pi)^6} \Gamma\left(\tfrac{1}{3}\right)^{18} ,$$

we easily deduce:

$$|\eta(j)| = \frac{3^{1/8}}{2\pi} \Gamma^{2/3}\left(\tfrac{1}{3}\right) ,$$

and this completes the proof of the Proposition. ∎

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