A Solution to the $\partial$–Problem for Holomorphic $(0,q)$–Forms, $q \geq 1$, on a Complex Normed Space

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1. INTRODUCTION

Using entirely elementary methods from differential calculus, we construct a $\mathcal{C}^\infty$–solution to the equation $\partial \omega = \omega$ where $\omega$ is a holomorphic $(0,q)$ form on a normed space or a Fréchet–Montel space or a DFM space. This extends in certain directions results in [4], [6] and [7]. This is in sharp contrast with the situation for the Cauchy–Riemann equation for $\mathcal{C}^\infty(0,q)$ forms $\omega$ where satisfactory solutions are only known for $q = 1$ on DFN spaces. Counterexamples in [1], [3] and [5] show that restrictions on the spaces, the coefficients and the degree are necessary. For $q > 1$, solutions are given for coefficients with polynomial growth in $L^2$ [4] and this is the only known result for arbitrary $q$. Solutions for $q = 1$ on separable Hilbert spaces and DFN spaces are given in [6]. The case $q = 2$ is solved with holomorphic coefficients in [7] and in [9] solutions are given for $(0,1)$ holomorphic forms on a Fréchet nuclear space.

2. THE CONSTRUCTION OF A SOLUTION OF $\partial \omega = \omega$ FOR HOLomorphic $(0,q)$ FORMS, $q \geq 1$, ON A NORMED SPACE

Let $E$ and $F$ be complex normed spaces. Let $\mathbb{K}$ be the field of real or complex numbers. For a positive integer $q \geq 1$, $\mathcal{L}_\mathbb{K}(qE;F)$ is the normed vector space of continuous $q$–$\mathbb{K}$–linear mappings from $E$ into $F$. Let $\mathcal{L}_\mathbb{K}(qF;F) = F$. The conjugate space of $E$ will be denoted by $\bar{E}$. Let $\mathcal{L}_\mathbb{C}(qE;F)$ be the normed vector space of continuous $q$–anti–linear mappings from $E$ into $F$. The notations $\mathcal{L}_\mathbb{C}(qE;F) = \mathcal{L}_\mathbb{C}(q\bar{E};F) = \mathcal{L}(q\bar{E};F)$ will be used.

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We denote by $\mathcal{L}_K(\mathcal{E}E)$ the Banach space of continuous $q$-linear forms on $E$. The Banach space of continuous $q$-anti-linear forms on $E$ will be denoted by $\mathcal{L}_A(\mathcal{E}E)$. For $q \geq 1$, let $\Lambda^{(0,q)}(E)$ be the Banach space of continuous alternating forms on $E$. For $q = 0$ let $\Lambda^{(0,0)}(E) = \mathcal{L}_C(\mathcal{E}E) = \mathcal{C}$.

If $\omega : \Omega \to \Lambda^{(0,q)}(E)$ is a $\mathcal{C}^1(0,q)$ form on an open subset $\Omega$ of $E$ let $\omega' : \Omega \to \mathcal{L}_K(E, \Lambda^{(0,q)}(E))$ be its derivative. Let $[\bar{\partial}]\omega(z) \in \mathcal{L}(E, \Lambda^{(0,q)}(E))$ be the anti-linear component of $\omega'(z)$ and let $\partial\omega(z)$ be the alternating component of $\omega(z)$. We use without explicit mention the isometry between $\mathcal{L}(E, \mathcal{L}(E; F), \mathcal{L}(\mathcal{E}E; F))$ and $\mathcal{L}(\mathcal{E}E, \mathcal{L}(\mathcal{E}E; F))$, $q \geq 1$. For each fixed $z \in E$ and $q > 1$ define $\tau_q(z) : E^{q-1} \to E$ by $\tau_q(z)(h_1, \ldots, h_{q-1}) = (h_1, \ldots, h_{q-1}, z)$. For $q = 1$, $\tau_1(z) = z$. For further details we refer to [7].

We now state some remarks which we use in the proof of our main result. The proofs are straightforward.

**Remark 2.1.** Let $T \in \Lambda^{(0,q)}(E) \to \mathcal{L}(\mathcal{E}E)$, $q \geq 2$. For each fixed $z \in E$, the function $T_1 : (E)^{q-1} \to \mathcal{C}$ defined by $T_1(z_1, \ldots, z_{q-1}) = T_1(z, z_1, \ldots, z_{q-1})$ is a continuous alternating $(q-1)$ anti-linear form on $E$. We write $T_1 = T(z)$. In particular, for a $(0, q)$ form $\omega$ on $E$, we have that $u : E \to \Lambda^{(0,q-1)}(E)$ defined by $u(z) = \omega(z)(z)$ is a $(0, q-1)$ form on $E$.

**Remark 2.2.** Let $\omega : \Omega \to \mathcal{L}_K(\mathcal{E}E; F)$, $q \geq 1$, be a $\mathcal{C}^1$ mapping. For $z$ in $\Omega$ and $h$ in $E$, $\lim_{t \to 0} (\omega(z + th) - \omega(z))/t = \omega'(z)h \in \mathcal{L}_K(\mathcal{E}E; F)$. Continuity of $\omega$ at $z$ implies that $\omega(z + th)$ tends to $\omega(z)$ in $\mathcal{L}_K(\mathcal{E}E; F)$ as $t \to 0$. For each fixed $z \in E$ the mapping $T \in \mathcal{L}_K(\mathcal{E}E; F) \to T \circ \tau_q(z) \in \mathcal{L}(\mathcal{E}E; F)$ is continuous. Hence it follows easily that $A : \Omega \to \mathcal{L}_K(\mathcal{E}E; F)$ defined by

$$A(z)(h_1, \ldots, h_{q-1}) = \omega(z) \circ \tau_q(z)(h_1, \ldots, h_{q-1}, z) = \omega(z)(h_1, \ldots, h_{q-1}, z)$$

is $\mathcal{C}^1$ and

$$A'(z) = \omega'(z) \circ \tau_{q+1}(z) + \omega(z) \circ \tau_q$$

**Remark 2.3.** Let $\omega : \Omega \to \mathcal{L}_K(\mathcal{E}E; F)$, $q \geq 3$, be a $\mathcal{C}^1$ mapping. For all $k = 0, 1, \ldots, q-2$, the mapping $B_k : \Omega \to \mathcal{L}_K(\mathcal{E}E; F)$ defined by

$$B_k(z)(h_1, \ldots, h_q) = \omega(z)(h_1, \ldots, h_k) \circ \tau_{q-k}(h_{k+1}, h_{k+2}, \ldots, h_q)$$

$$= \omega(z)(h_1, \ldots, h_k, h_{k+2}, \ldots, h_q, h_{k+1})$$

is $\mathcal{C}^1$ and
\[ B_k(z)(h_1, \ldots, h_{n+1}) = [\omega'(z)(h)(h_1, \ldots, h_k, h_{k+1}, \ldots, h_q, h_{k+1}) \]
\[ = \omega'(z)(h, h_1, \ldots, h_k) \circ \tau_{q-k}(h_{k+1})(h_{k+2}, \ldots, h_q). \]

Now, we have the following crucial lemma.

**Lemma 2.4.** Let \( \omega : \Omega \to L^{(2E); F} \) be a \( \mathcal{F}^\infty \) mapping. Define \( A : \Omega \to L^{R(E); F} \) by \( A(z) = \omega(z) \circ \tau(z) \). Then \( A \) is a \( \mathcal{F}^\infty \) mapping and for \( n \geq 1 \)

\[ A^{(n)}(z) = \omega^{(n)}(z) \circ \tau_{n+2}(z) + \sum_{k=0}^{n-1} B_k(z). \]

Where \( B_k(z)(h_1, \ldots, h_{n+1}) = \omega^{(n-1)}(z)(h_1, \ldots, h_k) \circ \tau_{n+1-k}(h_{k+1})(h_{k+2}, \ldots, h_{n+1}). \)

**Proof.** We prove this result by induction. Let \( n = 1 \). \( A : \Omega \to L^{R(E), L(E); F} \equiv L^{(2E); F} \). By applying (1) (in remark 2.2) we have

\[ A^{(n)}(z) = \omega^{(n)}(z) \circ \tau_{n+2}(z) + \omega(z) \circ \tau_{n+1} \]
\[ = \omega^{(n)}(z) \circ \tau_{n+2}(z) + \omega^{(n-1)}(z) \circ \tau_{n+1} + B_0(z). \]

Suppose (3) is true for \( n \geq 1 \).

We consider \( B : \Omega \to L^{R(n+1); F} \) defined by \( B(z) = \omega^{(n)}(z) \circ \tau_{n+2}(z). \)

Since \( \omega^{(n)} : E \to L^{R(n+2); F} \) is \( \mathcal{F}^1 \), \( B \) is \( \mathcal{F}^1 \) and

\[ B'(z) = \omega^{(n+1)}(z) \circ \tau_{n+2}(z) + \omega^{(n)}(z) \circ \tau_{n+2}. \]

On the other hand the mapping \( B_k : \Omega \to L^{R(n+1); F} \) for \( k = 0, 1, \ldots, n-1 \) is \( \mathcal{F}^1 \) and \( B_k : \Omega \to L^{R(n+2); F} \) is given by

\[ B_k(z)(h_1, \ldots, h_{n+1}) = \omega^{(n)}(z)(h_1, \ldots, h_k) \circ \tau_{n+1-k}(h_{k+1})(h_{k+2}, \ldots, h_{n+1}) \]
\[ = \omega^{(n)}(z)(h_1, \ldots, h_k, h_{k+2}, \ldots, h_{n+1}, h_{k+1}). \]

Hence \( A \) is a \( \mathcal{F}^{n+1} \) mapping and

\[ A^{(n+1)}(z) = \omega^{(n)}(z) \circ \tau_{n+3}(z) + \omega^{(n)}(z) \circ \tau_{n+2}(z) + \sum_{k=0}^{n-1} B_k'(z). \]

Let \( C_k : \Omega \to L^{R(n+2); F} \), \( k = 1, \ldots, n \), be defined by

\[ C_k(z)(h_1, \ldots, h_{n+2}) = \omega^{(n)}(z)(h_1, \ldots, h_k) \circ \tau_{n+2-k}(h_{k+1})(h_{k+2}, \ldots, h_{n+2}). \]
Define $C_0(z) = \omega^{(n+1)}(z) \circ \tau_{n+2}$. For $k = 1, \ldots, n$, we have the following equalities

$$
\omega^{(n+1)}(z)(h_1, \ldots, h_k) \circ \tau_{n+k-2}(h_{k+2}, \ldots, h_{n+2}) = \omega^{(n)}(z)(h_1, \ldots, h_k) \circ \tau_{n+k-1}(h_{k+2}, \ldots, h_{n+2})
$$

$$
= \omega^{(n)}(z)(h_1, \ldots, h_k) \circ \tau_{n+k-1}(v_1, \ldots, v_{k-1}) \circ \tau_{n+1-k-1}(v_k, v_{k+1}, \ldots, v_{n+1})
$$

(where $v_{j-1} = h_j$ for all $j$, $2 \leq j \leq n+2$).

These equalities together with (5) imply that

$$
C_k(z) = D_{k-1}^e(z).
$$

The last equality together with (6) imply that

$$
A^{(n+1)}(z) = \omega^{(n+1)}(z) \circ \tau_{n+2}(z) + \sum_{k=0}^n C_k(z).
$$

This proves (3) for $n+1$ and by induction this complete the proof.

Now, we are in a position to prove our main result.

**Theorem 2.5.** Let $E$ be a complex normed space and let $\Omega$ be an open subset of $E$. Let $\omega : \Omega \to \Lambda^{(0,q)}(E)$ be a holomorphic $(0,q)$ form on $\Omega$, $q \geq 1$. If $u : \Omega \to \Lambda^{(0,\bar q-1)}(E)$ is defined by $u(z) = \omega(z)(z)$, then $u$ is a $\mathcal{S}^\infty$ $(0,q-1)$ form and $\overline{\partial} u = \omega$ on $\Omega$.

**Proof.** By the inclusion $\Lambda^{(0,q)}(E) \hookrightarrow \mathcal{L}^q(E)$, we may suppose that $\omega : \Omega \to \mathcal{L}^q(E)$ is a $\mathcal{S}^\infty$ mapping. In fact $\omega : \Omega \to \mathcal{L}^q(E, \mathcal{L}^{q-1}(E))$ and $u : \Omega \to \mathcal{L}^q(E)$. By (1), $u$ is $\mathcal{S}^1$ and

$$
u(z) = \omega(z) \circ \tau_2(z) + \omega(z).
$$

Now, $\omega'$ can be seen as the following $\mathcal{S}^\infty$ mapping

$$
\omega' : \Omega \to \mathcal{L}_R(E, \mathcal{L}_R(\bar q - 1)(E)) \equiv \mathcal{L}_R^q(E, \mathcal{L}_R(\bar q - 1)(E)).
$$

Consider the mapping

$$
A : \Omega \to \mathcal{L}_R(E, \mathcal{L}_R(\bar q - 1)(E)) \equiv \mathcal{L}_R^q(E)
$$

defined by $A(z) = \omega'(z) \circ \tau_2(z) = \omega'(z)(z) = \omega'(z)(\cdot)(z)$.

By applying lemma 2.4, $A$ is a $\mathcal{S}^\infty$ mapping. Hence, (7) shows that $u$ is $\mathcal{S}^\infty$ on $\Omega$. From (7) it follows easily that
\[ [\partial]u(z)(y) = [\partial]\omega(z)(y, z) + \omega(z)(y) \quad \text{for } z \in \Omega \text{ and } y \in E. \]

Since \( \omega \) is holomorphic, \( \omega'(z) \in \mathcal{L}_E(E, \mathcal{L}(\mathcal{E}^E)) \) and hence \([\partial]\omega(z)(y, z) = 0\) for all \( y \in E \). Hence, (8) can be written as

\[ [\partial]u(z) = \omega(z). \]

Since \( \omega(z) \) is an alternating \( q \)-anti-linear form, we have \( \partial u = \omega \).

This complete the proof of theorem 2.5.

We now solve the \( \bar{\partial} \)-problem for holomorphic \((0, q)\) forms, \( q \geq 1 \), on Fréchet–Montel and DFM spaces.

**Theorem 2.6.** Let \( E \) be a complex DFM or a complex Fréchet–Montel space and let \( \omega : E \rightarrow \Lambda^{(0,q)}(E) \), \( q \geq 1 \), be a holomorphic \((0, q)\) form on \( E \). If \( u : E \rightarrow \Lambda^{(0,q-1)}(E) \) is defined by \( u(z) = \omega(z)(z) \), then \( u \) is a \( \mathcal{E}^\infty \) \((0,q-1)\) form on \( E \) and \( \partial u = \omega \) on \( E \).

**Proof.** a) Let \( E \) be a DFM space. By a result of Colombeau and Mujica [2] on factorization of holomorphic mappings from a DFM space into a metrizable locally convex space, we have for holomorphic \( \omega : E \rightarrow \Lambda^{(0,q)}(E) \), \( q \geq 1 \), that there exists a convex, balanced, open subset \( U \) of \( E \) such that \( \omega \) factors as in diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\omega} & \Lambda^{(0,q)}(E) \\
\pi_U & \downarrow & \downarrow p_U \\
E_U & \xrightarrow{\bar{\partial}u} & \Lambda^{(0,q-1)}(E_U)
\end{array}
\]

where \( E_U \) is the normed space associated with \( U \), \( \pi_U \) is the canonical map, \( p_U(A)(y_1, \ldots, y_q) = \hat{A}(\hat{y}_1, \ldots, \hat{y}_q) \), \( \pi_U(y_j) = \hat{y}_j \) and \( \bar{\partial}u = \hat{\omega} \) is a holomorphic mapping of bounded type, i.e., bounded on the balls of \( E_U \).

Theorem 2.5 implies that \( \tilde{u} : E \rightarrow \Lambda^{(0,q-1)}(E_U) \) defined by \( \tilde{u}(\tilde{z}) = \hat{\omega}(\hat{z})(\hat{\tilde{z}}) \) is \( \mathcal{E}^\infty \). In fact, \( \tilde{u} \) is a \((0,q-1)\) form of bounded type and \( \bar{\partial}u = \tilde{\omega} \) on \( E_U \). Hence a \( \mathcal{E}^\infty \) \((0,q-1)\) form \( u : E \rightarrow \Lambda^{(0,q-1)}(E) \) can be defined such that \( u \) is of uniform bounded type, i.e., the following diagram commutes

\[
\begin{array}{ccc}
E & \xrightarrow{u} & \Lambda^{(0,q-1)}(E) \\
\pi_U & \downarrow & \downarrow p_U \\
E_U & \xrightarrow{\tilde{u}} & \Lambda^{(0,q-1)}(E_U)
\end{array}
\]
and $\bar{\partial}u = \omega$ on $E$. We refer to [7] (Section 5) for further details. It is easy to see that $u(z) = \omega(z)(z)$.

b) Now, we prove theorem 2.6 for a Fréchet–Montel space $E$. Following arguments given in [9] (lemma 3.3) we can show that a holomorphic $(0,q)$ form $\omega$ on $E$ can be locally factorized through some $E_U$, i.e., for every $z \in E$ there exists a convex, balanced, open subset $U$ of $E$ such that $\omega$ factors as in the diagram

\[
x + U \subset E \quad \xrightarrow{\omega} \Lambda^{(0,q)}(E_U)
\]

\[
\xrightarrow{\pi_U}
\]

\[
\hat{x} + \hat{U} \subset E_U \quad \xrightarrow{\hat{\omega}_x} \Lambda^{(0,q)}(E_U)
\]

where $\hat{\omega}_x$ is a holomorphic $(0,q)$–form on $\hat{x} + \hat{U}$.

By theorem 2.5, $\hat{u}_x : \hat{x} + \hat{U} \subset E_U \rightarrow \Lambda^{(0,q-1)}(E_U)$ defined by $\hat{u}_x(\hat{x}) = \hat{\omega}_x(\hat{x})(\hat{x})$ is $\mathcal{O}^\infty$ and $\bar{\partial}\hat{u}_x = \hat{\omega}_x$ in $\hat{x} + \hat{U}$.

Now $u : E \rightarrow \Lambda^{(0,q-1)}(E)$ defined by $u(z) = \omega(z)(z)$ factors as in the following diagram

\[
x + U \subset E \quad \xrightarrow{u} \Lambda^{(0,q-1)}(E_U)
\]

\[
\xrightarrow{\pi_U}
\]

\[
\hat{x} + \hat{U} \subset E_U \quad \xrightarrow{\hat{u}_x} \Lambda^{(0,q-1)}(E_U)
\]

In particular, $u$ is $\mathcal{O}^\infty$, $u$ factorizes locally through some $E_U$ and $\bar{\partial}u = \omega$ on $E$. Theorem 2.6 extends to holomorphic $(0,q)$ forms, results given in [7], for holomorphic $(0,2)$ forms on DFN spaces, and results in [9], for holomorphic $(0,1)$ forms on Fréchet nuclear spaces.

References


9. **SORAGGI, R.L.**, A global solution to the $\bar{\partial}$–problem for holomorphic $(0,1)$ forms on a Fréchet nuclear space, manuscript.