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# The Spatial Flatness and Injectiveness of Connes Operator Algebras

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let $A$ be a Banach operator algebra, acting on a Banach space $E$. In this case $E$ is naturally considered as a Banach left $A$-module with the outer multiplication $a.x = a(x)$ where $a$ is an operator in $A$, and $x \in E$. A typical problem of the homological theory of operator algebras is to describe, for given $A$, homological properties of this particular $A$-module.

We say, for brevity, that $A$ has some homological property spatially, if $E$ has just this property. For example, * spatially flat operator algebras are those with flat $E$, etc.

Among the existing results in this direction, it is known that algebras of all bounded and of all compact operators on a arbitrary Banach space $E$ are spatially projective (Kaliman and Selivanov, [12]), and that nest algebras in a Hilbert space are spatially flat (Golovin [3]). Recently Golovin gave a complete characterization of spatially projective and spatially flat nondecomposable CSL – algebras [4].

In the present paper, we concentrate on the case of operator $C^*$-algebras, that is self-adjoint and norm closed subalgebras of the algebra $\mathbf{B}(H)$ of all bounded operators on a Hilbert space $H$, and specially on the "classical" case of von Neumann algebras (i.e. those operator $C^*$-algebras, which are weakly-operator closed in $\mathrm{B}(H)$). As to spatially projective von Neumann algebras, their characterization in inner terms of their structure was given in [8]. It turned out, that the assumption of the spatial projectivity is rather strong. In particular, all von Neumann algebras with such a property must belong to the

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type I, but the converse is false : for example, every infinite factor in the standar form is not spatially projective.

In what follows we show that, in the framework of von Neumann algebras, spatially flat, and also spatially injective algebras, form much larger classes than that of spatially projective algebras. In particular all *Connes' algebras* (that is hyperfinite, or equivalently, "Connes injective" von Neumann algebras) have both properties. Thus, all algebras of type I, and many other algebras, are spatially flat as well as spatially injective.

Proceeding to strict formulation, we shall recall briefly several standar notions of the "Banach" homology theory (for more details see e.g. [5] or [6, Ch. VII]).

Let $A$ be a (so far arbitrary) Banach algebra. As usual, we denote the categories of Banach left modules, Banach right modules, and Banach bimodules respectively by $A-$mod, mod$-A$, and $A-$mod$-A$.

If $X$, $Y \in A-$mod (respectively, mod$-A$, $A-$mod$-A$), then the space of morphisms between $X$ and $Y$ in the relevant category is denoted by ${}_A h(X,Y)$ (respectively, $h_A(X,Y)$, ${}_A h_A(X,Y)$). An additive functor from any of this categories to the category Ban of Banach spaces is called exact if it sends every admissible (= splitting in Ban) sequence to an exact sequence.

A Banach left (right, bi$-$) module $X$ is called *flat* if the tensor product functor $? \hat{\otimes}_A X : \text{mod}-A \longrightarrow \text{Ban}$ (respectively, $X \hat{\otimes}_A ? : A-\text{mod} \longrightarrow \text{Ban}$, $? \hat{\otimes}_{A-A} X = X \hat{\otimes}_{A-A} ? : \text{mod}-A \longrightarrow \text{Ban}$) is exact. (The bimodule tensor product $Y \hat{\otimes}_{A-A} X$, $X$, $Y \in A-$mod$-A$, can be defined as $Y \hat{\otimes}_{A^{\text{env}}} X$, where $A^{\text{env}}$ is the enveloping Banach algebra of $A$, $X$ is considered as a Banach left, and $Y$ as a Banach right module over $A^{\text{env}}$. It is the same Banach space as $X \hat{\otimes}_{A^{\text{env}}} Y$).

A Banach left (right, bi$-$) module over $A$ is called *injective* if the morphism functor ${}_A h(?,X) : A-\text{mod} \longrightarrow \text{Ban}$ (respectively, $h_A(?,X) : \text{mod}-A \longrightarrow \text{Ban}$, ${}_A h_A(?,X) : A-\text{mod}-A \longrightarrow \text{Ban}$) is exact. From the definitions of the flatness and of the injectivity, one can easily deduce that a *Banach left (right, bi$-$) module $X$ is flat if and only if its dual right (respectively left, bi$-$) module $X^*$ is injective.*

Now let $A$ be a Banach operator algebra on a Hilbert space $H$. For a dual Banach left (right) $A-$module $X$ and fixed $x \in X$ consider the operator $m_x : A \longrightarrow X$; $a \longrightarrow a.x$ ($m_x{}^{\circ} : A \longrightarrow X$; $a \longrightarrow x.a$). In the spirit of Kadison and Ringrose [10], we say that a dual Banach left (right, bi$-$) module $X$ is *normal* if $m_x$ (respectively, $m_x{}^\circ$, both $m_x$ and $m_x{}^\circ$) is (are) normal, that is ultra weakly–weak* continuous, for every $x \in X$.

At last, let $A$ be an operator C*–algebra on $H$, and $A^-$ be its ultraweak (or equivalently, weak–operator) closure in $\mathbf{B}(H)$. Such an $A$ is called Connes amenable algebra, or just *Connes algebra* (cf. [1]), if all continuous derivations of $A$ with values in normal $A$–bimodules are inner. (It follows from Theorems 1 and 2 in [7] that such a property implies that every continuous derivation of $A$ with values in every dual $A$–bimodule, normal as a left or right module, is also inner.) As a combined result of Johnson–Kadison–Ringrose [9] and Connes [1] (with a contribution of Elliot [2]), $A$ is a Connes algebra if and only if $A^-$ is hyperfinite, or equivalently, is "Connes injective" (that is, there exists a projeection $\mathbf{B}(H) \longrightarrow A^-$ of norm 1).

Sometimes we shall use the words "one–sided module" as a common name for left and right modules. Now we are able to formulate the main result of this paper.

THEOREM. *Every normal one–sided module over a Connes algebra is injective. If in addition it is reflexive as a Banach space, then it is also flat.*

The proof will given in a separate section. At the moment, taking the theorem for granted, we discuss some of its corollaries.

It is obvious that, for every $x \in H$, the operator $\mathbf{B}(H) \longrightarrow H$ ; $a \longrightarrow a(x)$ is weak operator–weak* continuous and hence normal. Thus, the theorem implies

COROLLARY 1. *Every Connes algebra is spatially flat and spatially injective.*

The theorem can be translated to the language of the cohomology groups. Recall (see e.g. [5, Ch. 0. § 4.2]) that, for a Banach algebra $A$ and $X$, $Y \in A$–mod (mod–$A$), the space $\mathbf{B}(X, Y)$ of all bounded operators between $Y$ and $X$ is considered as a Banach $A$–bimodule with outer multiplications defined as $[a.\varphi](x) = a.(\varphi(x))$ and $[\varphi.a](x) = \varphi(a.x)$ (respectively, $[a.\varphi](x) = \varphi(x.a)$ and $[\varphi.a](x) = (\varphi(x)).a)$; $a \in A$, $\varphi \in \mathbf{B}(Y, X)$, $x \in Y$. It is known that the following properties of $X$ are equivalent: a) $X$ is injective; b) $H^1(A, \mathbf{B}(Y, X)) = 0$ for all $Y \in A$–mod (mod–$A$); and c) $H^n(A, \mathbf{B}(Y, X)) = 0$ for all $Y \in A$–mod (mod–$A$) and $n > 0$. Taking into account this, and also the obvious isomorphism $\mathbf{B}(Z, X^*) \simeq \mathbf{B}(X, Z^*)$ where $X \in A$–mod (mod–$A$) is reflexive as a Banach space and $Z \in$ mod–$A$ ($A$–mod), we immediately have

COROLLARY 2. *Let $A$ be a Connes algebra on a Hilbert space $H$, and $X$ be a normal left (right) $A$-module. Then $H^n(A, \mathbf{B}(Y,X)) = 0$ for all $Y \in A$*-mod (mod-$A$) *and $n > 0$. If in addition $X$ is reflexive as a Banach space, then $H^n(A, \mathbf{B}(X,Z^*)) = 0$ for all $Z \in$* mod-$A$ ($A$-mod) *and $n > 0$. In particular, $H^n(A, \mathbf{B}(H,Z^*)) = 0$ for all $Z \in$* mod-$A$ *and $n > 0$ (cf.* [8, *Corollary*]*).*

## 2. THE PROOF OF THE MAIN THEOREM

We begin with several preparatory facts. In what follows, the duality between a Banach space and its dual will be always denotes by $<.,.>$.

Let $Y \in A$-mod (mod-$A$), $Z \in A$-mod-$A$. In this situation the morphism space ${}_Ah(Y,Z)$ (respectively $h_A(Y,Z)$) is considered as a Banach right (left) $A$-module with the outer multiplication defined by $[\varphi.a](x) = (\varphi(x)).a)$ (respectively, $[a.\varphi](x) = a.(\varphi(x))$; cf. [5, Ch. 0. 4. 2]. As to the following lemma, we feel that it is well-known to specialists (on the "folklore" level). Nevertheless, we shall give its proof for the completeness of our exposition.

LEMMA 1. *If, with $Y$ and $Z$ as before, $Z$ is an injective $A$-bimodule, then ${}_Ah(Y,Z)$ (respectively, $h_A(Y,Z)$) is an injective right (respectively, left) $A$-module.*

*Proof.* Without the loss of generality, we can assume that $Y \in A$-mod: the argument in the case $Y \in$ mod-$A$ is strictly parallel to what follows.

As it is known [5, Ch. III.1.4], a Banach (bi) module is injective if and only if it is a retract of a so called cofree (bi) module in the respective category. In particular, that means that $Z$ is a retract of an $A$-bimodule of the form $U := \mathbf{B}(A_+ \hat{\otimes} A_+, E)$, where $E$ is a Banach space, $A_+$ is $A$ with the adjoint identity, and outer multiplication are well defined by $[a.\varphi](b \otimes c) := \varphi(b \otimes ca)$ and $[\varphi.a](b \otimes c) = \varphi(ab \otimes c)$; $a \in A$; $b, c \in A_+$; $\varphi \in U$. In virtue of functorial properties of ${}_Ah(Y,?)$, this implies that ${}_Ah(Y,Z)$ is a retract of a right $A$-module $V := {}_Ah(Y,U)$.

Now notice that $V$ is isomorphic to the right $A$-module $W := \mathbf{B}(A_+, \mathbf{B}(X,E))$ with the outer multiplication defined by $[\varphi.a](b) := \varphi(ab)$ ; $a \in A$; $b \in A_+$; $\varphi \in W$. Indeed, it is easy to check that maps $\alpha : V \longrightarrow W : \psi \longrightarrow \mathscr{X}$, where $\mathscr{X}$ is defined by $[\mathscr{X}(a)] := [\psi(x)](a \otimes e)$, and $\beta : W \longrightarrow V : \mathscr{X} \longrightarrow \psi$, where $\psi$ is well defined by $[\psi(x)](a \otimes b) := [\mathscr{X}(a)](b.x)$, are mutually inverse morphisms in mod-$A$ (here $a, b \in A_+$, $x \in \mathscr{X}$, and $e$ is the adjoint identity in $A_+$). Hence ${}_Ah(Y,Z)$ is a retract of a module $\mathbf{B}(A_+,F)$ (with $F=\mathbf{B}(X,E)$), that is of a cofree Banach right $A$-module. It follows from what was said above that it is injective. ∎

LEMMA 2. *Let $A$ be a Banach algebra with the identity, and $X$ be a Banach one-sided $A$-module with zero outer multiplication. Then $X$ is injective.*

*Proof.* Again, we can restrict ourselves with the case of a left module $X$. Consider the cofree Banach left $A$-module $U:=\mathbf{B}(A_+,X)$ with the outer multiplication defined by $[a.\varphi](b)=\varphi(ba)$; $a\in A$, $b\in A_+$, $\varphi\in U$, and the map $\alpha: X\longrightarrow U: x\longrightarrow\varphi$, where $\varphi(a)=a.x$. It is obvious that $\alpha$ is a morphism in $A$-mod. Further, consider the map $\beta: U\longrightarrow X:\varphi\longrightarrow\varphi(e_+-e)$, where $e$ is the identity in $A$ and $e_+$ is the adjoint identity in $A_+$. Since

$$\beta(a.\varphi)=a.\varphi(e_+-e)=\varphi(e_+a-ea)=\varphi(a-a)=0=a.\beta(\varphi),$$

$\beta$ is also a morphism in $A$-mod, and it is obvious that it is a left inverse to $\alpha$. That means that $X$ is a retract of a cofree module, and therefore (cf. the proof of Lemma 1) it is injective. ∎

LEMMA 3. *Let $A$ be a Banach operator algebra on a Hilbert space, and let $X=(X_*)^*$ be a reflexive normal one-sided $A$-module. Then its dual $X^*$ is also normal.*

*Proof.* Again, we can assume that $X$ is a left module. Since it is normal, we have $\lim_\nu<a_\nu.x,y>=<a.x,y>$ for every net $a_\nu\in A$; $\nu\in\Lambda$, ultraweakly converging to some $a\in A$, and for every $y\in X_*$. But since $X$ is reflexive, it implies that $\lim_\nu<f,a_\nu.x>=<f,a.x>$ or, equivalently, $\lim_\nu<f,a_\nu.x>$ $=<f,a.x>$ for every $f\in X^*$ and $x\in X=(X^*)_*$. Thus $X^*$ is a normal right $A$-module. ∎

*Continuation of the proof of the Theorem.* Suppose $X=(X^*)_*$ is a normal left module over a Connes algebra $A$. We begin with the principal case, when $A$ has an identity, say $e$, and $X$ is unital (that is, $e.x=x$ for all $x\in X$).

Consider the dual $A$-bimodule $A^*$ of $A$ and observe that $X$ is isomorphic in $A$-mod to $h_A(X_*,A^*)$ with the outer multiplication described in the begining of this section. Indeed, it is easy to check that $\alpha: x\longrightarrow\varphi$, where $\varphi$ is defined by $<\varphi(y),a>=<a.x,y>$ and $\beta:\varphi\longrightarrow x$, where $<x,y>=<\varphi(y),e>$; $a\in A$; $x\in X$; $y\in X_*$, are mutually inverse morphisms of these modules.

Now consider the closed sub$-A-$bimodule $A_*^-$ of $A^*$ consisting of normal functionals. (In virtue of Sakai's theorem, it is just the predual of $A^-$, see e.g. [13].) Choose an arbitrary $\varphi \in h_A(X_*, A^*)$ and $y \in X_*$, and put $f := \varphi(Y)$; suppose that a net $a_\nu \in A; \nu \in \Lambda$ ultraweakly converges to some $a \in A$. Then

$$<f, a> = <\varphi(y).a, e> = <\varphi(y.a), e> = <e.x, y.a> = <a.x, y> \ ,$$

and similarly $<f, a_\nu> = <a_\nu.x, y>$ for all $\nu \in \Lambda$. Since $X$ is normal, $\lim_\nu <a_\nu.x, y> = <a.x, y>$ and $\lim_\nu f(a_\nu) = f(a)$. That means $f$ is normal. Hence $\varphi(y) \in A_*^-$ for every $\varphi \in h_A(X_*, A^*)$ and $y \in X_*$. Therefore $h_A(X_*, A^*) = h_A(X_*, A_*^-)$, and $X$ is isomorphic to the latter module.

But it was proved in [7, Theorem 3] that an operator C*$-$algebra $A$ is a Connes algebra if and only if $A_*^-$ is an injective $A-$bimodule. Therefore it follows from Lemma 1 that $X$ is injective.

Proceed to the case when $A$ has an identity, but $X$ is not unital. Then $X$ canonically decomposes in $A-$mod into the direct sum of its submodules $X_u = \{x \in X : e.x = x\}$ and $X_0 = \{x \in X : e.x = 0\}$. Since $X_u$ is obviously unital and normal (with the predual $\{y \in X_* : y.e = y\}$, it follows from what was said above that it is injective. Since $X_0$ has zero outer multiplication, it is injective by virtue of Lemma 2. Hence $X$ is a direct sum of two injective modules, and therefore it is injective itself.

Assume, at last, that $A$ has no identity. Denote by $e$ the projection in $\mathbf{B}(H)$ with the image $A.H$, the closure of the linear span of the set $\{a(x) : a \in A, x \in H\}$, and put $A_\times := A \oplus \{\lambda e : \lambda \in \mathbb{C}\}$. Obviously, $A_\times$ is an operator C*$-$algebra with the identity $e$ (topologically isomorphic to the "abstract" unitalization $A_+$ of $A$). Moreover, since Connes algebras can be completely described in terms of their ultra$-$weak closures (cf. the introduction), and $(A_\times)^-$ obviously coincides with $A^-$, we conclude that $A_\times$ is also a Connes algebra.

As in the standard process of unitalization, every $Y \in A-$mod becomes a Banach left unital $A_\times-$module with the outer multiplication $b.x := a.x + \lambda x$, where $b = a + \lambda e$; $a \in A$; $\lambda \in \mathbb{C}$; the latter will be denoted by $Y_\times$. Every morphism of $A-$modules becomes, after this identification, a morphism of $A_\times-$modules. Moreover, $Y$ is normal, the same is true to $Y_\times$ : it obviously follows from the observation that a net $b_\nu = a_\nu + \lambda_\nu e$; $\nu \in \Lambda$ ultraweakly converges to $b = a + \lambda e$; $a_\nu, a \in \Lambda$; $\lambda_\nu, \lambda \in \mathbb{C}$ if and only if $a_\nu$ ultraweakly converges to $a$, and $\lambda_\nu$ converges to $\lambda$. In particular, $X_\times$ is normal as well as $X$, and it follows from what was proved in the "unital" case, that $X_\times$ is injective in $A_\times$ -mod.

It remains to check the definition of injectivity for $X$. Take an admissible complex $\mathcal{Y}$ of Banach left $A$-modules; it follows from what was said above that it can be considered as an admissible complex in $A_\times$-mod, say $\mathcal{Y}_\times$. Since $X_\times$ is injective in $A_\times$-mod, the complex ${}_{A_\times}h(\mathcal{Y}_\times, \mathcal{X}_\times)$ of Banach spaces is exact. It follows that the complex ${}_A h(\mathcal{Y}, \mathcal{X})$, which just coincide with the latter, is also exact. This shows that $X$ is injective.

Now the first assertion of the theorem is proved. Combining it with Lemma 3, we have the following: if $X$ is normal and reflexive, then $X^*$ is injective. But, as it was mentioned in the introduction, it is just to say that $X$ is flat.

We gave the complete proof of the theorem for the case of left modules. The same argument, with obvious modifications, is valid for right modules. ∎

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