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Supertauberian Operators and Perturbations[1]

M. GONZÁLEZ AND A. MARTÍNEZ-ABEJÓN

Dpto. de Matemáticas, Fac. de Ciencias, 39071 *Santander, Spain*
Dpto. de Matemáticas, Fac. de Ciencias, 33007 *Oviedo, Spain*

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Upper semi–Fredholm operators and tauberian operators on Banach spaces (definitions below) admit the following perturbative characterizations [6],[2]: An operator $T: X \longrightarrow Y$ is upper semi–Fredholm (tauberian) if and only if for every compact operator $K: X \longrightarrow Y$ the kernel $N(T+K)$ is finite dimensional (reflexive).

In [7] Tacon introduces an intermediate class between upper semi–Fredholm operators and tauberian operators, the supertauberian operators, and he studies this class using non–standard analysis.

Here we study supertauberian operators using ultrapower of Banach spaces and, among other results, we obtain a perturbative characterization. As a consequence we characterize Banach spaces in which all superreflexive subspaces are finite dimensional, and Banach spaces in which all reflexive subspaces are superreflexive. Similar results are obtained for the dual class of cosupertauberian operators, including a perturbative characterization of this class, and characterizations of Banach spaces in which all quotients are finite dimensional, and Banach spaces in which all reflexive quotients are superreflexive.

We use standard notations: X and Y are Banach spaces, B_X the closed unit ball of X, S_X the set of elements of norme one of X, $\mathcal{B}(X,Y)$ the class of bounded linear operators from X to Y, X^* the dual of X, $T^*: Y^* \longrightarrow X^*$ the adjoint of $T \in \mathcal{B}(X,Y)$, $R(T)$ and $N(T)$ the range and the kernel of T. We identify X with a subspace of X^{**}. We denote by $\mathbf{N}$ the set of all positive integers.

An operator $T \in \mathcal{B}(X,Y)$ is said to be upper semi–Fredholm if $R(T)$ is closed and $N(T)$ is finite dimensional; and it is said to be tauberian if it satisfies $T^{**-1}(Y) \subset X$. Tauberian operators were introduced by Kalton and Wilansky [5].

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Given a number $\epsilon \in (0,1)$, an operator $T \in B(X,Y)$ ia said to be an ϵ-isometry if $|\,\|Tx\| - \|x\|\,| < \epsilon\|x\|$ for all $x \in X$. A Banach space X is said to be finitely representable in Y if for each $\epsilon > 0$ and every finite dimensional subspace M of X there exists an ϵ-isometry $T\colon M \longrightarrow Y$.

We denote $\ell_\infty(X)$ the Banach space of bounded sequences (x_i) in X with norm $\|(x_i)\| := \sup\|x_i\|$.

Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbf{N}$ (which be fixed along the paper), and let $N(\mathcal{U},X)$ be the closed subspace of all sequences $(x_i) \in \ell_\infty(X)$ which converge to 0 following $\mathcal{U}$. The ultrapower of X (following $\mathcal{U}$) is defined as follows:

$$X_{\mathcal{U}} := \ell_\infty(X) \,/\, N_{\mathcal{U}}(X).$$

The element of $X_{\mathcal{U}}$ including as a representative the sequence $(x_i) \in \ell_\infty(X)$ is denoted by $[x_i]$. The norm of $[x_i]$ in $X_{\mathcal{U}}$ is given by $\|[x_i]\| = \lim_{\mathcal{U}}\|x_i\|$.

The ultrapower $X_{\mathcal{U}}$ contains an isometric copy of X generated by the constant sequences of $\ell_\infty(X)$. We identify this copy with X. An operator $T \in B(X,Y)$ admits an extension $T_{\mathcal{U}} \in B(X_{\mathcal{U}}, Y_{\mathcal{U}})$ which is given by $T_{\mathcal{U}}[x_i] = [Tx_i]$.

The ultrapower $(X^*)_{\mathcal{U}}$ is contained isometrically in $(X_{\mathcal{U}})^*$, but in general they do not coincide. Therefore, if $(f_i) \in \ell_\infty(X^*)$, then $[f_i]$ is identified with an element of $(X_{\mathcal{U}})^*$. Heinrich [3] proves that $(X^*)_{\mathcal{U}}$ coincides with $(X_{\mathcal{U}})^*$ if and only if X is superreflexive. We refer to [3] for basic results about ultrapowers of Banach spaces.

1. Supertauberian Operators

Recall that a Banach space X is said to be superreflexive if each space Z finitely representable in X is reflexive. We use the next characterization of superreflexivity established by James [4].

THEOREM 1. [4] *A Banach space X is superreflexive if and only if for every real number $\epsilon \in (0,1)$ there is a positive integer n for which there do not exist finite sets $\{x_1,\dots,x_n\}$ in S_X and $\{f_1,\dots,f_n\}$ in S_{X^*} for which $f_j(x_i) > \epsilon$ for $1 \leqslant j \leqslant i \leqslant n$, and $f_j(x_i) = 0$ for $1 \leqslant i < j \leqslant n$.*

Tacon [7] call $T \in B(X,Y)$ supertauberian if for all number $\epsilon \in (0,1)$ there exists a positive integer n for which there do not exist finite sets $\{x_1,\dots,x_n\}$ in S_X and $\{f_1,\dots,f_n\}$ in S_{X^*} for which $f_j(x_i) > \epsilon$ for $1 \leqslant j \leqslant i \leqslant n$, $f_j(x_i) = 0$ for $1 \leqslant i < j \leqslant n$, and $\|Tx_k\| < 1/k$ for $k = 1,2,\dots,n$.

Next we give a useful characterization of supertauberian operators.

PROPOSITION 2. *An operator $T \in B(X,Y)$ is supertauberian if and only if for every $\epsilon \in (0,1)$ there exists $\delta > 0$ and a positive integer n for which there do not exist finite sets $\{x_1,\dots,x_n\}$ in X and $\{f_1,\dots,f_n\}$ in X^* for which $f_j(x_i) > \epsilon$ for $1 \leqslant j \leqslant i \leqslant n$, and $f_j(x_i) = 0$ for $1 \leqslant i < j \leqslant n$.*

It is an immediate consequence of the definition that the kernel of a supertauberian operator is superreflexive. Next we prove a stronger result.

PROPOSITION 3. *If $T \in B(X,Y)$ is supertauberian then $N(T_{\mathcal{U}})$ is superreflexive.*

The next two results are necessary in the proof of the main theorem (theorem 10).

PROPOSITION 4. *If $T \in B(X,Y)$ and if $N(T_{\mathcal{U}})$ is reflexive, then $N(T^{**}) \subset X$.*

THEOREM 5. *For an operator $T \in B(X,Y)$ the following assertions are equivalent.*

a) T is supertauberian.

b) $T_{\mathcal{U}}$ is tauberian.

c) $N(T_{\mathcal{U}})$ is reflexive.

COROLLARY 6. *If $T \in B(X,Y)$ is supertauberian and $K \in B(X,Y)$ is compact, then $T+K$ is supertauberian.*

The following result is similar to one of the characterizations of superreflexive Banach spaces, due to James. It will be essential for the proof of proposition 8 and theorem 10.

PROPOSITION 7. *An operator $T \in B(X,Y)$ is supertauberian if and only if there exists $\epsilon \in (0,1)$, $\delta > 0$ and $n \in \mathbf{N}$ for which there do not exist finite sets $\{x_1,\dots,x_n\}$ in X and $\{f_1,\dots,f_n\}$ in X^* for which $f_j(x_i) > \epsilon$ for $1 \leqslant j \leqslant i \leqslant n$, and $f_j(x_i) = 0$ for $1 \leqslant i < j \leqslant n$, and $\|Tx_k\| < \delta$ for $k = 1,\dots,n$.*

Next result is proved by Tacon [7] using non-standard analysis. An easier proof can be given using proposition 7.

PROPOSITION 8. *The class of supertauberian operators is open in $B(X,Y)$.*

For a supertauberian operator T, corollary 6 implies $N(T+K)$ is superreflexive for every compact operator K. To obtain the converse, given $T \in$

$B(X,Y)$ no supertauberian, we need to find a compact operator K such that $N(T+K)$ is not superreflexive. Since $T_{\mathcal{U}}$ is not tauberian, by [2, theorem 1] there exists a compact operator $S \in B(X_{\mathcal{U}}, Y_{\mathcal{U}})$ such that $N(T_{\mathcal{U}}+S)$ is not reflexive. Unfortunately not all the operators in $B(X_{\mathcal{U}}, Y_{\mathcal{U}})$ are $K_{\mathcal{U}}$ for some $K \in B(X,Y)$. Furthermore, supposing there is an operator $K \in B(X,Y)$ such that $K_{\mathcal{U}} = S$, if $R(T+K)$ is not closed then $(N(T+K))_{\mathcal{U}}$ is strictly contained in $N(T_{\mathcal{U}}+K_{\mathcal{U}})$. In this way, it may occur that $N(T+K)$ is superreflexive although $N(T_{\mathcal{U}}+K_{\mathcal{U}})$ is not reflexive. We will obtain K in a direct way, using no ultrapowers, as an application of the following technical lemma.

LEMMA 9. *Let* $\epsilon \in (0,1)$, *and let* $\{x_1,\dots,x_n\}$ *and* $\{f_1,\dots,f_n\}$ *be finite sets in* S_X *and* S_{X^*} *respectively, such that* $f_j(x_i) > \epsilon$ *for* $1 \leqslant j \leqslant i \leqslant n$, *and* $f_j(x_i) = 0$ *for* $1 \leqslant i < j \leqslant n$. *Then there is a finite set* $\{g_1,\dots,g_n\}$ *in* X^* *such that* $g_i(x_j) = \delta_{ij}$ *and* $\|g_i\| \leqslant \epsilon^{-1}(1+\epsilon^{-1})^{n-1}$ *for every* i.

THEOREM 10. *An operator* $T \in B(X,Y)$ *is supertauberian if and only if for every compact operator* $K \in B(X,Y)$ *the kernel* $N(T+K)$ *is superreflexive.*

COROLLARY 11. *If* $T \in B(X,Y)$ *is not supertauberian, then there exists a compact operator* $K \in B(X,Y)$, *with* $\|K\|$ *arbitrarily small, such that* $N(T+K)$ *is not superreflexive.*

As an application of theorem 10 we will characterize some classes of Banach spaces.

PROPOSITION 12. *Let* X *be a Banach space.*

a) Superreflexive subspaces of X *are finite dimensional if and only if every supertauberian operator from* X *into any Banach space is upper semi–Fredholm.*

b) Reflexive subspaces of X *are superreflexive if and only if every tauberian operator from* X *into any Banach space is supertauberian.*

Situations described in the above theorem are not trivial. For instance, the original Tsirelson space T^* [1, VI.a.1] is reflexive no superreflexive, and any of its infinite dimensional subspaces or quotients contain an isomorphic copy of T^*. Thus every $T \in B(T^*,Y)$ is tauberian; but it is supertauberian if and only if it is upper semi–Fredholm. Moreover, if X is an infinite dimensional superreflexive space, then there are operators $T \in B(X,Y)$ which are not upper semi–Fredholm, but all of them are supertauberian.

Recall that an operator $K \in B(X,Y)$ is said to be super weakly compact if for all number $\epsilon > 0$ there exists a positive integer n for which there do not exist finite sets $\{x_1,\ldots,x_n\}$ in S_X and $\{f_1,\ldots,f_n\}$ in S_{Y^*} for which $f_j(Kx_i) > \epsilon$ for $1 \leqslant j \leqslant i \leqslant n$, and $f_j(Kx_i) = 0$ for $1 \leqslant i < j \leqslant n$.

We have (see [3]) that K is super weakly compact if and only if $K_{\mathcal{U}}$ is weakly compact. Next, using this class of operators, we give an algebraic characterization of supertauberian operators.

THEOREM 13. *For $T \in B(X,Y)$, the following assertions are equivalent:*

a) The operator T is supertauberian.

b) For every Z and $A \in B(Z,X)$, if TA is super weakly compact then A is super weakly compact.

2. COSUPERTAUBERIAN OPERATORS

Recall [8] that $T \in B(X,Y)$ is said to be cotauberian (cosupertauberian) if T^* is tauberian (supertauberian). In this section we will give for cosupertauberian operators analogous results to that of the previous section for supertauberian operators. We observe that it will be not always possible to use duality to derive the results.

The proof of the next result can be obtained using duality from proposition 8.

PROPOSITION 14. *The class of cosupertauberian operators is open in $B(X,Y)$.*

Next we give the main result of this section.

THEOREM 15. *An operator $T \in B(X,Y)$ is cosupertauberian if and only if $Y/\overline{R(T+K)}$ is superreflexive for all compact operator $K \in B(X,Y)$.*

As an application of theorem 15 we will characterize some class of Banach spaces. Recall that an operator $T \in B(X,Y)$ is said to be lower semi–Fredholm if $R(T)$ is closed and $Y/R(T)$ is finite dimensional.

THEOREM 16. *Let Y be a Banach space.*

a) Superreflexive quotients of Y are finite dimensional if and only if every cosupertauberian operator from any Banach space into Y is lower semi–Fredholm.

b) Reflexive quotients of Y are superreflexive if and only if every cotauberian operator from any Banach space into Y is cosupertauberian.

As in theorem 12, situations described in the previous theorem are not trivial. For instance, if T^* is the Tsirelson space, then every cosupertauberian $T \in B(X,T^*)$ is lower semi–Fredholm, but not every $T \in B(X,T^*)$ is cosupertauberian.

Finally we present an algebraic characterization of cosupertauberian operators.

THEOREM 17. *For $T \in B(X,Y)$, the following assertions are equivalent:*

a) The operator T is cosupertauberian.

b) For every Z and $A \in B(Y,Z)$, if AT is super weakly compact then A is super weakly compact.

REFERENCES

1. CASAZZA, P.G. AND SHURA, T.J., "Tsirelson Space", Lecture Notes in Math. 1363, Springer Verlag, 1989.
2. GONZÁLEZ, M. AND ONIEVA, V.M., Characterizations of tauberian operators and other semigroups of operators, *Proc. Amer. Math. Soc.* **108** (1990), 399–405.
3. HEINRICH, S., Ultraproducts in Banach space theory, *J. reine angew. Math.* **313** (1980), 72–104.
4. JAMES, R.C., Some self–dual properties of normed linear spaces, in "Sympos. on Infinite Dimensional Topology", Ann. of Math. Studies 69, Princeton Univ. Press, Princeton, 1972, 159–175.
5. KALTON, N. AND WILANSKY, A., Tauberian operators on Banch spaces, *Proc. Amer. Math. Soc.* **57** (1976), 251–255.
6. LEBOW, A. AND SCHECHTER, M., Semigroups of operators and measures of non–compactness, *J. Funct. Anal.* **7** (1971), 1–26.
7. TACON, D.G., Generalized semi–Fredholm transformations, *J. Austral Math. Soc.* (Series A) **34** (1983), 60–70.
8. TACON, D.G., Generalized Fredholm transformations, *J. Austral Math. Soc.* (Series A) **37** (1984), 89–97.