EXTRACTA MATHEMATICAE Vol. 8, Núm. 2–3, 153 – 157 (1993)

Positive Solutions of Nonlinear Delay Integral Equations Modelling Epidemics and Population Growth

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AMS *Subject Class.* (1980): 45G10, 92A15, 58C30 *Received November* 2, 1993

In this paper we present some of the results of [3],[4] concerning the existence of nontrivial and nonnegative solutions of integral equations of the type

$$x(t)=\int_0^{\tau(t)} g(t,s,x(t-s-l))\,\mathrm{d}s \tag{1}$$

where the following set of assumptions is supposed in the whole paper:

(H) $g:\mathbb{R}\times\mathbb{R}\times[0,+\infty)\longrightarrow\mathbb{R}$, $(t,s,y)\longrightarrow g(t,s,y)$ is a continuous function, ω–periodic $(\omega>0)$ in the variable t, $g(t,s,y)\geqslant 0$, $\forall(t,s,y)\in\mathbb{R}\times\mathbb{R}\times[0,+\infty)$, $g(t,s,0)=0$, $\forall(t,s)\in\mathbb{R}\times\mathbb{R}$, l is a nonnegative constant and $\tau:\mathbb{R}\longrightarrow\mathbb{R}^+$ is a continuous and λ–periodic function $(\lambda>0)$ such that $\omega/\lambda=p/q$, $p,q\in\mathbb{N}$.

Equations of type (1) have been considered as a model to explain the evolution of some infectious diseases and have been also used as a growth equation for single species population when the birth rate varies seasonally. It includes, as a particular case, different eqautions considered by other authors (see [2],[7],[8],[9],[11],[12] and [13]). Specially, we must remark the model elaborated by Cooke and Kaplan [7] where $\tau(t)\equiv\tau>0$, $l=0$ and g is of the form $g(t,s,y)=f(t-s,y)$; in this case the quantity $a(t-s)=\partial f(t-s,0)/\partial y$ (if this derivative exists) may be interpreted as the contact rate of the epidemy. The fact that g is ω–periodic in t means that the contact rate of the epidemy is ω–periodic in t which usually happens in the practice; on the other hand, as $g(t,s,0)=0$, $\forall(t,s)\in\mathbb{R}\times\mathbb{R}$, $x(t)\equiv 0$ is always a solution of (1).

Due to the origin of problem (1) it would be of a great interest to have a

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[1] This work was supported by DGICYT, Ministry of Education and Science (Spain), under grant number PB92–0941.

[2] This paper was done whereas the second author was visiting the Department of Mathematical Analysis of the University of Granada. He thanks this Department for its hospitality.

good information about different questions such as the existence and nonexistence of nonnegative and nontrivial $q\omega$–periodic solutions, uniqueness, multiplicity, stability, etc...

Several authors have studied particular cases of (1). Respect to the existence of solutions, basically, three types of techniques have been used: topological methods (essentially, the fixed point index; see [7],[11]), bifurcation methods (see [9]) and monotone methods (upper and lower solutions; see [2],[8]). The uniqueness of nonnegative and nontrivial solutions of (1) is treated in [10] and [11]. Some multiplicity results have been obtained in [8],[13]. To the best of our knowledge none interesting result is known about the stability of solutions of (1).

In what follows we present two results whose detailed proof may be seen in [3],[4]. After each one, we comment which are the main novelties respect to the mentioned bibliography.

E will denote the real Banach space of all real and continuous $q\omega$–periodic function defined on $\mathbb{R}$, with the norm

$$\|x\| = \max_{0 \leqslant t \leqslant q\omega} |x(t)|, \ \forall x \in E.$$

Also, $P = \{x \in E : x(t) \geqslant 0, \ \forall t \in \mathbb{R}\}$. If $L : E \longrightarrow E$ is a linear and continuous operator, $r(L)$, the spectral radius of L, is defined as

$$r(L) = \lim_{n \to \infty} \|L^n\|^{1/n}.$$

THEOREM 1. ([3]) *Let us suppose the following hypotheses:*

(H_1) *There exists a positive (automatically continuous) function* $a : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\lim_{y \to 0^+} \frac{g(t,s,y)}{y} = a(t,s)$$

uniformly in $(t,s) \in \mathbb{R} \times \mathbb{R}$.

(H_2) *There exists a function* $b : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\lim_{y \to +\infty} \frac{g(t,s,y)}{y} = b(t,s)$$

uniformly in $(t,s) \in \mathbb{R} \times \mathbb{R}$.

Then if

$$r(L(\tau,a)) > 1 \ \text{ and } \ r(L(\tau,b)) < 1, \tag{2}$$

equation (1) *has a solution in* $P \setminus \{0\}$, *where* $L(\tau,a) : E \longrightarrow E$ *is defined by*

$$(L(\tau,a)x)(t) = \int_0^{\tau(t)} a(t,s)\, x(t-s-l)\, ds, \ \forall x \in E,$$

(analogously for $L(\tau,b)$*).*

Moreover if $b(t,s) > 0$, $\forall (t,s) \in \mathbb{R}\times\mathbb{R}$ *and*

$$b(t,s)y < g(t,s,y) < a(t,s)y, \ \forall y > 0, \ \forall (t,s) \in \mathbb{R}\times\mathbb{R}, \tag{3}$$

then (2) *is also necessary for the existence of a solution of* (1) *in* $P\backslash\{0\}$.

For the proof, we need to set problem (1) as a fixed point problem of the type

$$x = Fx$$

in an appropriate ordered Banach space (E, respect to the order induced by the cone P) and then, by using the fixed point index (see [1] for a good review of the subject) we prove the existence of solutions of (1) in $P\backslash\{0\}$.

Remarks. 1) In [7],[9],[11] the function b is identically zero, so that our previous theorem allows g to have a better behaviour at infinity respect to the variable y.

2) To the best of our knowledge it is the first time that a condition as (2) is shown to be necessary for that (1) has a solution in $P\backslash\{0\}$.

3) Different applications of the previous theorem to some concrete situations may be seen in [3].

THEOREM 2. ([4]) *Let us assume the following:*

(H_1) *There exists a continuous function* $a: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\liminf_{y\to 0^+} \frac{g(t,s,y)}{y} = a(t,s)$$

uniformly in $(t,s) \in \mathbb{R}\times\mathbb{R}$.

(H_2) *There exists a continuous function* $b: \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\limsup_{y\to +\infty} \frac{g(t,s,y)}{y} = b(t,s)$$

uniformly in $(t,s) \in \mathbb{R}\times\mathbb{R}$.

(H_3) g *is increasing with respect to* $y \in [0,+\infty)$ *for fixed* $(t,s) \in \mathbb{R}\times\mathbb{R}$.

Then if

$$\min_{t\in\mathbb{R}} \int_0^{\tau(t)} a(t,s)\, ds > 1 \ \text{and} \ \max_{t\in\mathbb{R}} \int_0^{\tau(t)} b(t,s)\, ds < 1, \tag{4}$$

equation (1) *has a solution* $x \in P\backslash\{0\}$, *which is strictly positive for all values of* t.

The proof uses an iterative monotone scheme to obtain the solution: from the hypotheses of the theorem one may find constants α and β such that $0<\alpha<\beta$ and

$$\alpha \leqslant \int_0^{\tau(t)} g(t,s,\alpha)\,\mathrm{d}s,\ \ \beta \geqslant \int_0^{\tau(t)} g(t,s,\beta)\,\mathrm{d}s,\ \ \forall t \in \mathbb{R}$$

(α and β are called, respectively, a lower and a upper solution of (1)). Then defining

$$x_1 = \alpha,\ \ x_{n+1}(t) = \int_0^{\tau(t)} g(t,s,x_n(t-s-l))\,\mathrm{d}s,\ \ \forall n \in \mathbb{N},\ \forall t \in \mathbb{R}$$

$$x^1 = \beta,\ \ x^{n+1}(t) = \int_0^{\tau(t)} g(t,s,x^n(t-s-l))\,\mathrm{d}s,\ \ \forall n \in \mathbb{N},\ \forall t \in \mathbb{R}$$

it is not difficult to prove that $\{x_n\}$ and $\{x^n\}$ converge, respectively, to functions x and y which are positive solutions of (1) (in general, it is not known if x and y are different functions).

Remarks. 1) Hypotheses (H_1) and (H_2) of Theorem 2 are, respectively, more general than (H_1) and (H_2) of Theorem 1. Also, in Theorem 2 we do not suppose that a is a positive function. However, (4) is less general than (2) and (H_3) does not appear in Theorem 1. The conclusion is that Theorems 1 and 2 are independent.

2) The main novelty of Theorem 2 is that we allow the function a to be zero for some values of (t,s). That is, if we are talking about the epidemic model, the epidemy may recurs periodically (even in a strictly positive way for all values of the time) although the contact rate may be zero on some intervals of time (it is clear that this may happens in the applications due to different causes, for example, to be on school holidays). A concrete example of this fact may be seen in [4]. Some nontrivial extensions of the previous results to a system of equations like (1) (a problem whose interest was already shown in [7]) may be seen in [5],[6].

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