Miščenko’s Theorem for Bitopological Spaces

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AMS Subject Class. (1991): 54E55, 54E35

Received September 28, 1993

In 1945, V.E. Šneider proved that every Hausdorff compact space with a $G_δ$-diagonal is metrizable [12]. By using Šneider’s theorem, M. Katětov stated in 1948 that a Hausdorff compact space $(X, T)$ is metrizable if and only if the space $(X\times X\times X, T\times T\times T)$ is hereditarily normal [4] and A. Miščenko proved in 1962 his celebrated theorem that a Hausdorff compact space with a point–countable base is metrizable [8].

Bitopological extensions of the theorems of Šneider and Katětov have been obtained in [9] and [10]. We here obtain the following generalization of Miščenko’s theorem: A pairwise Hausdorff pairwise countably compact bitopological space $(X, P, Q)$ is quasi–metrizable if and only if both $P$ and $Q$ have a point–countable base. Actually, we will prove a more general result in terms of point–countable $T_1$–separating open covers. (Further results about pairwise compact quasi–metrizable spaces may be found in [6] and [7]).

Let us recall some definitions.

A quasi–metric on a set $X$ is a non–negative real–valued function $d$ on $X\times X$ such that for all $x, y, z \in X$, i) $d(x, y) = 0 \iff x = y$; ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Each quasi–metric $d$ on $X$ induces a $T_1$ topology $T(d)$ on $X$ which has as a base the family $\{B_d(x, r) : x \in X, r > 0\}$ where $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

Note that if $d$ is a quasi–metric on $X$, then the function $d^{-1}(x, y) = d(y, x)$ for each $x, y \in X$, is also a quasi–metric on $X$ called conjugate of $d$.

Bitopological spaces appear in a natural way when one considers the topologies $T(d)$ and $T(d^{-1})$ naturally associated with a quasi–metric and its

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1 The authors thank the support of the DGICYT grant PB89–0611.
conjugate. A bitopological space is [5] and ordered triple \((X,P,Q)\) such that \(X\) is a nonempty set and \(P\) and \(Q\) are topologies on \(X\).

A bitopological space \((X,P,Q)\) is called:

i) pairwise regular if for all \(x \in X\), the \(Q\)-closed \(P\)-neighborhoods of \(x\) form a base for the \(P\)-neighborhoods of \(x\) and \(P\)-closed \(Q\)-neighborhoods of \(x\) form a base for the \(Q\)-neighborhoods of \(x\) [5].

ii) pairwise Hausdorff if for \(x \neq y\) there is a \(P\)-neighborhood of \(x\) and a disjoint \(Q\)-neighborhood of \(y\).

iii) quasi-metrizable if there is a quasi-metric \(d\) on \(X\) such that \(T(d) = P\) and \(T(d^{-1}) = Q\).

In [2], P. Fletcher, H.B. Hoyle III and C.W. Patty introduced the notion of a pairwise compact bitopological space and proved that every pairwise Hausdorff pairwise compact space is pairwise regular. In the following we will make use of a useful characterization of pairwise (countably) compact spaces due to M.K. Singal and R. Singal [11] (see also [1]):

A bitopological space \((X,P,Q)\) is pairwise (countably) compact if and only if every proper \(P\)-closed subset is \(Q\)-(countably) compact and every proper \(Q\)-closed subset is \(P\)-(countably) compact.

The letter \(\mathbb{N}\) will denote the set of positive integers. If \(P\) is a topology for a set \(X\) and if \(A \subseteq X\), we write \(\text{cl}_P A\) for the closure of \(A\) in \((X,P)\).

We will also use the following auxiliary results:

**Lemma 1.** ([9]) A pairwise countably compact space \((X,P,Q)\) such that each proper countably compact subspace of \((X,P)\) has a \(G_\delta\)-diagonal is pairwise compact.

**Lemma 2.** ([9]) Let \((X,P,Q)\) be a pairwise Hausdorff pairwise countably compact space and let \(P\) be second countable. Then \((X,P,Q)\) is quasi-metrizable and \(Q\) is second countable.

Recall that a topological space \((X,T)\) has countable pseudo-character if for each \(x \in X\), \(\{x\} = \cap \{V_n(x) : n \in \mathbb{N}\}\) where each \(V_n(x)\) is an open set.

**Lemma 3.** Let \((X,P,Q)\) be a pairwise Hausdorff pairwise countably compact space such that \((X,P)\) has a \(G_\delta\)-diagonal and \((X,Q)\) has countable pseudo-character. Then \((X,P,Q)\) is quasi-metrizable.

**Proof.** By Lemma 1, \((X,P,Q)\) is pairwise compact. Fix \(x \in X\). Then \(\{x\} = \cap \{V_n(x) : n \in \mathbb{N}\}\) where each \(V_n(x)\) is an open set.
∩{V_n(z) : n ∈ N} where each V_n(z) is Q-open. So X \ {z} = ∪{X \ V_n(z) : n ∈ N}.

Take a point in X, y ≠ z, and let {y} = ∩{V_n(y) : n ∈ N} where each V_n(y) is Q-open. Then there is a k ∈ N such that z ∈ X \ V_k(y). Thus X = (X \ V_k(y)) \ (∪{X \ V_n(z) : n ∈ N}). Now X \ V_k(y) is P-compact since it is a proper Q-closed subset of X. Similarly, each X \ V_n(z) is P-compact. Therefore (X,P) is σ-compact and, consequently, it is a Lindelöf space. The quasi-metrizability of (X,P,Q) follows from the fact [9, Theorem 3] that a pairwise Hausdorff pairwise countably compact space (X,P,Q) is quasi-metrizable if (X,P) is a Lindelöf space with a G_δ-diagonal.

A cover ℘ of a set X is called T_1-separating if for each x ∈ X, \{x\} = ∩{C ∈ ℘ : x ∈ C ∈ ℘} [3].

**Theorem.** A pairwise Hausdorff pairwise countably compact space (X,P,Q) is quasi-metrizable if and only if (X,P) has a point-countable T_1-separating open cover and (X,Q) has countable pseudo-character.

**Proof.** Sufficient condition: Let ℘ be a point-countable T_1-separating open cover for (X,P). For each P-countably compact subset F of X put

℘ \cap F = \{C \cap F : C ∈ ℘\}.

Then ℘ \cap F is a point-countable T_1-separating open cover of the subspace (F,P|F). Since a countably compact topological space has a G_δ-diagonal if and only if it has a point-countable T_1-separating open cover [3, p. 475], (F,P|F) has a G_δ-diagonal. By Lemma 1, (X,P,Q) is pairwise compact.

Now fix z ∈ X. Then \{z\} = ∩{V_n(z) : n ∈ N} where each V_n(z) is Q-open. Since (X,P,Q) is pairwise regular, for each n ∈ N there is a Q-open set W_n(z) such that z ∈ W_n(z) ⊆ cl_P W_n(z) ⊆ V_n(z). Thus \{z\} = ∩{cl_P W_n(z) : n ∈ N}. But X \ W_n(z) is P-countably compact, so that the subspace

(X \ W_n(z),P|X \ W_n(z))

has a G_δ-diagonal as we have shown above. By Lemma 3, the subspace

(X \ W_n(z),P|X \ W_n(z),Q|X \ W_n(z))

is quasi-metrizable and by [9, Theorem 1], P|X \ W_n(z) is second countable. Then P|X \ cl_P W_n(z) is also second countable. Now take a point in X, y ≠ z; thus \{y\} = ∩{cl_P W_n(y) : n ∈ N} where each W_n(y) is Q-open, so that there is a k ∈ N such that z ∈ X \ cl_P W_k(y). Since P|X \ cl_P W_k(y) is second countable, P is
second countable. By Lemma 2, \((X, P, Q)\) is quasi-metrizable.

Necessary condition: It follows immediately from the fact that if \((X, P, Q)\) is a pairwise Hausdorff pairwise countably compact quasi-metrizable space then both \(P\) and \(Q\) are second countable [9, Theorem 1].

**Corollary.** A pairwise Hausdorff pairwise countably compact space \((X, P, Q)\) is quasi-metrizable if and only if both \((X, P)\) and \((X, Q)\) have a point-countable base.

**Proof.** If both \((X, P)\) and \((X, Q)\) have a point-countable base, then \((X, P)\) has a point-countable \(T_1\)-separating open cover and \((X, Q)\) has countable pseudo-character. \(\blacksquare\)

Note that if in the above corollary we put \(P = Q\) one obtains Miščenko’s theorem.

**References**