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A Proof of the Markov–Kakutani Fixed Point Theorem Via the Hahn–Banach Theorem

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S. Kakutani, in [2] and [3], provides a proof of the Hahn–Banach theorem via the Markov–Kakutani fixed point theorem, which reads as follows.

THEOREM. *Let K be a compact convex set in a locally convex Hausdorff space E. Then every commuting family $(T_i)_{i\in I}$ of continuous affine endomorphisms on K has a common fixed point.*

In this note I wish to point out how to obtain, conversely, this theorem from the Hahn–Banach theorem. The use of the Hahn–Banach theorem necessitates formulating the Markov–Kakutani theorem in the setting of locally convex spaces. Actually, it holds in a general Hausdorff topological vector space as well and has a well–known and simple proof (see e.g. [1, p. 456]); but it is applied for the most part in locally convex spaces, for instance in order to show the existence of Haar measure on a compact abelian group. So the point of this note is rather to illustrate the power of the Hahn–Banach theorem than to simplify the proof of the Markov–Kakutani theorem.

The key of the proof of the theorem lies on the following lemma, which of course is a special case of the Schauder–Tychonov fixed point theorem. However, its assumptions are strong enough to allow a completely elementary treatment. It is here that the Hahn–Banach theorem, in the form of the separation theorem, enters.

LEMMA. *Let K be a compact convex set in a locally convex Hausdorff space E, and let $T: K \longrightarrow K$ be a continuous affine transformation. Then T has a fixed point.*

Proof. If the lemma were false, the intersection of the diagonal $\Delta := \{(x,x):$

$x \in K\}$ of $K \times K$ with the graph of T, viz. $\Gamma := \{(x, Tx) : x \in K\}$, would be empty. Since Δ and Γ are compact convex subsets of $E \times E$, the Hahn–Banach theorem applies to produce continuous linear functionals l_1 and l_2 on E and numbers $\alpha < \beta$ such that

$$l_1(x) + l_2(x) \leqslant \alpha < \beta \leqslant l_1(y) + l_2(Ty)$$

for all $x, y \in K$. Consequently,

$$l_2(Tx) - l_2(x) \geqslant \beta - \alpha$$

for all $x \in K$. Iterating this inequality yields

$$l_2(T^n x) - l_2(x) \geqslant n(\beta - \alpha) \longrightarrow \infty$$

for arbitrary $x \in K$ so that the sequence $(l_2(T^n x))_{n \in \mathbb{N}}$ is unbounded, which contradicts the compactness of $l_2(K)$. ∎

The Markov–Kakutani theorem is now readily established by means of a simple compactness argument: Let K_i denote the set of all fixed points of T_i. We have $K_i \neq \emptyset$ by the lemma, and K_i is compact and convex. To show $\bigcap_{i \in I} K_i \neq \emptyset$, which is our aim, it is enough to do so for finite intersections. Since T_i and T_j commute, we conclude $T_i(K_j) \subset K_j$. Hence, $T_i|_{K_j}$ has a fixed point by the lemma so that $K_i \cap K_j \neq \emptyset$. An obvious induction argument now shows $\bigcap_{i \in F} K_i \neq \emptyset$ for all finite $F \subset I$.

References

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