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Maximal Plurisubharmonic Functions on Domains in Banach Spaces

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An important role in classical potential theory in $\mathbb{R}^n$ is played by the construction of harmonic functions on domains with prescribed boundary behaviour. The boundary condition required is usually continuity except at isolated points where singularities of certain kinds are sought e.g. when $n=2$, logarithmic singularities are frequently required and for $n>2$ singularities proportional to $(\text{distance})^{2-n}$ are of interest. A similar theory has been developed in complex potential theory for $\mathbb{C}^n$ using the Monge–Ampere operator in place of the Laplacian, but with logarithmic singularities in all dimensions ([1], [3], [4], [5], [6]). Using maximal plurisubharmonic functions in place of the Monge–Ampere operator, Lelong [5, 6] has extended this theory to Banach spaces with a finite number of logarithmic poles or singularities and in [1] the author discusses the infinite dimensional case for an infinite number of poles. In this article we consider the fine density of maximal plurisubharmonic functions at an accumulation point of these logarithmic poles. We refer to [2] and [4] for background information on plurisubharmonic functions on infinite dimensional Banach spaces.

An upper semicontinuous functions $f\colon \Omega \longrightarrow \mathbb{R} \cup \{-\infty\}$, where Ω is an open subset of a Banach space E is said to be *plurisubharmonic* if it is not identically equal to $-\infty$ and

$$f(x) \leqslant \frac{1}{2\pi}\int_0^{2\pi} f(x+yre^{i\theta})\,d\theta$$

for all $x \in \Omega$, $y \in E$ and r sufficiently small.

We let $PSH(\Omega)$ denote the set of all plurisubharmonic functions on Ω.

If U is a bounded open subset of Ω, $\overline{U} \subset \Omega$, $g \in PSH(\Omega)$ and $f \leqslant g$ on ∂U (the boundary of U) implies $f \leqslant g$ on U then we say that g is a *maximal plurisubharmonic function* on Ω.

If $\dim(E) = 1$ then f is maximal plurisubharmonic if and only if it is harmonic while if $\dim(E) < \infty$ and f is locally bounded, then f is maximal plurisubharmonic if and only if it satisfies the homogeneous (complex) Monge–Ampere equation.

The *Lelong density*, $v_f(a)$, of $f \in PSH(\Omega)$ at a point a in Ω is defined as

$$\liminf_{z \to a} \{f(z) / \log\|z - a\|\} .$$

If $\lim_{z \to a} \{f(z) / \log\|z - a\|\}$ exists then we call $v_f(a)$ the regular density of f at a.

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of points in the open set Ω. Furthermore, suppose that this sequence is bounded away from the boundary of Ω and that it has precisely one accumulation point, a_0, in Ω. We may in fact suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the point a_0 in Ω. Let $(v_n)_{n=1}^{\infty}$ denote a sequence of non-negative real numbers (weights) with $\Sigma_{n=1}^{\infty} v_n < \infty$.

The set $\Omega' := \Omega \setminus \{a_n\}_{n=0}^{\infty}$ is open and $\delta(\Omega') = \delta(\Omega) \cup \{a_n\}_{n=0}^{\infty}$.

The following result is proved in [5] for finite sequences and in [1] for sequences with the properties given above.

PROPOSITION 1. *If Ω is a bounded hyperconvex domain in a Banach space and $\{a_n\}_{n=0}^{\infty}$ is a sequence with the properties given above, then there exists $g \in PSH(\Omega)$ satisfying the following*:

(i) $g_{|\Omega'}$ *is a maximal plurisubharmonic function*,

(ii) $g_{|\Omega \setminus \{a_0\}}$ *is continuous*,

(iii) $g \leqslant 0$ *on* Ω,

(iv) $\limsup_{z \to \delta\Omega} g(z) = 0$,

(v) $v_g(a_n) = v_n$ *for* $n = 1, 2, \ldots$.

Remarks. (a) The function g is uniquely determined by the above properties, and is called the (*complex*) *Green function* with logarithmic poles of density v_n at a_n.

(b) A domain is called *hyperconvex* if it admits a negative plurisubharmonic exhaustion function. A function satisfying (iii) and (iv) is a negative *exhaustion function* and so hyperconvexity is a necessary condition for the proposition.

(c) The hypothesis that $(a_n)_n$ be bounded away from the boundary is necessary in order to obtain (iv).

(d) An important role in the proof of Proposition 1 is played by the function

$$l(z) := \Sigma_{n=1}^{\infty} v_n \log \|z - a_n\| \,.$$

This function is the only known general plurisubharmonic lower estimate for g, i.e. $l \leqslant g$. An example in [5] shows that in general we have $l < g$. If $\Sigma_{n=1}^{\infty} v_n = +\infty$ then $l(z) \equiv -\infty$ and, consequently, the requirement $\Sigma_{n=1}^{\infty} v_n < \infty$ is necessary until new nontrivial lower bounds for g are found.

(e) In general g does not extend continuously to Ω. For example if Ω is the ball of radius 2, $\|a_n\| = 1/n$ for $n \leqslant 1$ and $v_n = 1/n^2$ then

$$g(0) \geqslant l(0) = \Sigma_{n=1}^{\infty} v_n \log \|a_n\| = -\Sigma_{n=1}^{\infty} \{\log n \,/\, n^2\} > -\infty \,.$$

On the other hand

$$\liminf_{z \to 0} g(z) \leqslant \lim_{n \to \infty} g(a_n) = -\infty$$

and g is not continuous at the origin.

In fact g is continuous on Ω if and only if $g(a_0) = -\infty$. If $g(a_0) > -\infty$ then $v_g(a_0) = 0$. Further situations in which $v_g(a_0) = 0$ are given in [1]. At the point a_0 it is clear that g does not have a regular density and in this paper we show that it is possible that g has a fine density (Definition 3 below) at a_0.

The *fine* (or *pluri-fine*) *topology* on an open subset Ω of a Banach space is the weakest topology on Ω such that all plurisubharmonic functions on Ω are continuous. Clearly the fine topology is weaker then the given topology on Ω. The notion of a thin set is helpful in discussing the weak topology.

A subset G of Ω is said to be *thin* at $p \in \Omega$ if either $p \notin \overline{G}$ or there exists U open in Ω, $p \in U$, and $h \in PSH(\Omega)$ such that

$$\lim_{x \to p,\, x \in G} h(z) < h(p) \,.$$

A subset G of Ω is said to be thin if it is thin at every point of Ω. With this concept one easily sees that sets of the form G, with G^c (the complement of G in Ω) thin at p, are a neighbourhood basis at p for the fine topology.

Our next proposition is a generalization of the classical *Weiner criterion* to Banach spaces.

PROPOSITION 2. *Let Ω denote a bounded domain in a Banach space. Then $A \subset \Omega$ is thin at the point $z_0 \in \overline{A} \cap \Omega$ if and only if there exists a strictly decreasing* *null sequence of positive real numbers,* $(r_n)_{n=1}^{\infty}$, *such that* $\Sigma_{j=1}^{\infty} U_{A_j}(z_0) > -\infty$ *where*

(i) $A_j = A \cap \{r_j > \|z - z_0\| \geqslant r_{j+1}\}$,

(ii) $U_{A_j} := U_{A_j}^{\Omega} = \sup\{v : v \in PSH(\Omega),\ v < 0 \text{ on } \Omega,\ v \leqslant -1 \text{ on } A_j\}$.

Proof. If $\Sigma_{j=1}^{\infty} U_{A_j}(z_0) > -\infty$ then there exists $f_j \in PSH(\Omega)$, $f_j < 0$, $f_j \leqslant -1$ on A_j and $f_j(z_0) > U_{A_j}(z_0) - 2^{-j-1}$. It follows that $f_J = \Sigma_{j=J}^{\infty} f_j \in PSH(\Omega)$ and

$$f_J(z_0) > -2^{-J} + \Sigma_{j=1}^{\infty} U_{A_j}(z_0)\ .$$

If J is chosen sufficiently large then $f_J(z_0) > -1$. However, $\limsup_{\xi \to z_0, \xi \in A} f_J(\xi) \leqslant$ $\leqslant -1$ and A is thin at z_0.

Conversely, if A is thin at z_0, then there exists $h \in PSH(\Omega)$ such that

$$\limsup_{\xi \to z_0, \xi \in A} h_J(\xi) < h(z_0)\ .$$

Using a truncation argument it is possible to find $H \in PSH(\Omega)$ such that

$$\limsup_{\xi \to z_0, \xi \in A} H(\xi) = -\infty < H(z_0)$$

and, without loss of generality, we may suppose $H < 0$ on Ω. Now choose $(r_n)_n$, a strictly decreasing null sequence, such that

$$\sup\{H(\xi) : \|z_0 - \xi\| < r_j\} < -2^j\ .$$

Hence

$$2^{-j} H(\xi) < -1 \quad \text{on} \quad \{\xi : \|z_0 - \xi\| < r_j\} \cap A\ .$$

Since $A_j \subset \{\xi : \|z_0 - \xi\| < r_j\} \cap A$ this implies $U_{A_j} \geqslant 2^{-j} H$ on Ω and in particular $U_{A_j}(z_0) \geqslant 2^{-j} H(z_0)$.

Hence $\Sigma_{j=1}^{\infty} U_{A_j}(z_0) \geqslant H(z_0) > -\infty$ and this completes the proof. ∎

DEFINITION 3. If $h \in PSH(\Omega)$, Ω an open subset of a Banach space, and $p \in \Omega$ then the *fine-density* of h at p, $\bar{v}_h(p)$, is defined as

$$\bar{v}_h(p) = \underset{z \to p}{\text{fine limit}}\ \{h(z) / \log\|z - p\|\}\ .$$

We now show that it is possible for the (complex) Green function to have a fine density at the accumulation point of logarithmic poles. In our proposition we let $\|a_n\| = 1/n$, mainly for the sake of simplicity. It is possible to develop the method to include more general situations by combining hypothesis involving the a_n's and the v_n's. An examination of the proof shows that a careful balance has to be maintained in order to obtain a limit.

THEOREM 4. *Let* Ω *be a hyperconvex domain of diameter* 1 *in a Banach space* E, $A = \{(a_n, v_n)_n\}_n$, $a_n \in E$, $\|a_n\| = 1/n$ *for* $n \geqslant 2$, $v_n > 0$ *for all* n, $\Sigma_{n=2}^{\infty} v_n < \infty$ *and* $\Sigma_{n=2}^{\infty} v_n \log n = \infty$. *Let* g *denote the maximal plurisubharmonic function with logarithmic poles of density* v_n *at* a_n *for all* n.

Suppose also that for each n *we can find* $\beta_n, \delta_n \in \mathbb{R}^+$ *such that the following conditions are satisfied*:

(i) $\Sigma_{n=2}^{\infty} \beta_n \log n < \infty$,

(ii) $\beta_n \log \delta_n \longrightarrow -\infty$ as $n \longrightarrow \infty$,

(iii) $(v_n \log \delta_n) / \log n \longrightarrow 0$ as $n \longrightarrow \infty$.

Then $\bar{v}_g(0) = 0$.

Proof. Let $h(z) = \Sigma_{n=2}^{\infty} \beta_n \log \|z - a_n\|$. Clearly $h \in PSH(\Omega)$ and $h(0) = \Sigma_{n=2}^{\infty} \beta_n \log(1/n) > -\infty$ by assumption (i).

Let $E_0 = \bigcup_{n=2}^{\infty} B(a_n, \delta_n)$ where $B(a, r) = \{z \in E : \|z - a\| < r\}$. By conditions (i) and (ii), we can suppose, without loss of generality, that the balls in E_0 do not overlap. If $z \in E_0$, say $z \in B(a_m, \delta_m)$ then $h(z) \leqslant \beta_m \log \delta_m$ since Ω has diameter 1 and the terms in h are negative. Thus, by assumption (ii) $\lim_{z \to 0, z \in E_0} h(z) = -\infty < h(0)$ and the set E_0 is thin at 0.

To prove the result it suffices to show $\lim_{z \to 0, z \notin E_0} \{l(z) / \log \|z\|\} = 0$ where

$$l(z) = \Sigma_{m=2}^{\infty} v_m \log \|z - a_m\|.$$

Let $w_m = \left\{ z \in \Omega : \frac{1}{m + \frac{1}{2}} < \|z\| \leqslant \frac{1}{m - \frac{1}{2}} \right\}$.

Then

$$l(z) / \log \|z\| = v_m \{\log \|z - a_m\| / \log \|z\|\} + \\ + \Sigma_{n<m} v_n \{\log \|z - a_n\| / \log \|z\|\} + \Sigma_{n>m} v_n \{\log \|z - a_n\| / \log \|z\|\}.$$

If $z \in w_m$, $z \in E_0$ we get

$$v_m \{\log \|z - a_m\| / \log \|z\|\} \leqslant v_m \left\{ \log \delta_m / \log \left[\frac{1}{m + \frac{1}{2}} \right] \right\} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow 0$$

by (iii). Also

$$\Sigma_{n>m} v_n \{\log\|z - a_n\| \big/ \log\|z\|\} \leqslant \Sigma_{n>m} v_n \left\{\log\left[\tfrac{1}{m+\frac{1}{2}} - \tfrac{1}{n}\right] \big/ \log\left[\tfrac{1}{m-\frac{1}{2}}\right]\right\}$$

since $\|z - a_n\| \geqslant |\,\|z\| - \|a_n\|\,|$.
Since $n > m$ we have

$$\log\left[\tfrac{1}{m+\frac{1}{2}} - \tfrac{1}{n}\right] \big/ \log\left[\tfrac{1}{m-\frac{1}{2}}\right] \leqslant \log\left[\tfrac{1}{m+\frac{1}{2}} - \tfrac{1}{m+1}\right] \big/ \log\left[\tfrac{1}{m-\frac{1}{2}}\right] \leqslant 3$$

for m sufficiently large and so

$$\Sigma_{n>m} v_m \{\log\|z - a_n\| \big/ \log\|z\|\} \leqslant 3 \cdot \Sigma_{n>m} v_m \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty.$$

For $n < m$ we have

$$\Sigma_{n<m} v_m \{\log\|z - a_n\| \big/ \log\|z\|\} \leqslant \Sigma_{n<m} v_m \{\log(\|a_n\| - \|z\|) \big/ \log\|z\|\} \leqslant$$

$$\leqslant \Sigma_{n<m} v_m \left\{\log\left[\tfrac{1}{n} - \tfrac{1}{m-\frac{1}{2}}\right] \big/ \log\left[\tfrac{1}{m-\frac{1}{2}}\right]\right\}.$$

For $n < m$ we have

$$\log\left[\tfrac{1}{n} - \tfrac{1}{m-\frac{1}{2}}\right] \big/ \log\left[\tfrac{1}{m-\frac{1}{2}}\right] \leqslant 3.$$

Let $N(m)$ be an increasing sequence of real numbers $N(m) < m$, such that

$$w(m) := \log\left[\tfrac{1}{N(m)} - \tfrac{1}{m-\frac{1}{2}}\right] \big/ \log\left[\tfrac{1}{m-\frac{1}{2}}\right] \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty$$

and $N(m) \longrightarrow \infty$ as $m \longrightarrow \infty$ (e.g. $N(m) = \log m$).
Then

$$\Sigma_{n<m} v_m \{\log\|z - a_n\| \big/ \log\|z\|\} =$$

$$= \Sigma_{n\leqslant N(m)} v_n \{\log\|z - a_n\| \big/ \log\|z\|\} + \Sigma_{N(m)<n<m} v_n \{\log\|z - a_n\| \big/ \log\|z\|\} \leqslant$$

$$\leqslant \left[\Sigma_{n\leqslant N(m)} v_m\right] w_m + \Sigma_{N(m)<n<m} 3v_n$$

$$\longrightarrow 0 \text{ as } m \longrightarrow \infty.$$

Combining the above estimates we get $\lim_{z\to 0,\, z\notin E_0} \{l(z) \big/ \log\|z\|\} = 0$. Since $l \leqslant g$ this implies $\lim_{z\to 0,\, z\notin E_0} \{g(z) \big/ \log\|z\|\} = 0$ and, as we have shown that E_0

is thin at 0, it follows that $\bar{v}_g(0) = 0$.
This completes the proof. ∎

EXAMPLE 5. The hypotheses of Theorem 4 are satisfied by the following sequence (with n sufficiently large):

$$v_n = \{n(\log n)^2(\log\log n)\}^{-1}$$

$$\beta_n = \{n(\log n)^2(\log\log n)(\log\log\log n)^2\}^{-1}$$

$$\delta_n = \exp\{-n(\log n)^2(\log\log n)(\log\log\log n)^3\}\ .$$

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