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About the Inverse Operations on the Hyperspace of Nonlinear Monotone Operators

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1. INTRODUCTION

Recently many authors have studied many operations, which preserve convexity on the hyperspace of convex functions. Some of the operations, like the epigraphic addition (infimal–convolution) and the right epigraphic scalar multiplication, which correspond to sum of epigraphs and the scalar multiplication of epigraphs (see [16], section 5). This research was motivated by the basic role in optimization and in the study of variational problems. As is was treated by Mazure [10], study of epigraphic sum of convex functions can be interpreted in terms of the inverse addition of their subdifferentials, which are monotone operators.

In this paper, we present an exposition of the inverse operations on the hyperspace $\mathscr{M}(X)$ of monotone operators, which are the inverse sum and the inverse scalar multiplication.

In order to help the reader to understand the basic ideas, the relevant historical background is given in what follows.

In studying the parallel connection of two resistors in an electrical network, Erickson was led to introduce a couple of dual operations called parallel addition. In order to consider also the electrical connection of multiports, Anderson and Duffin [1] extended these operations from the scalar case to the case in which the operators are symmetric positive definite matrices. Fillmore and Williams [9] extended these operations to the class of bounded positive operators on a Hilbert space. Anderson, Morly and Trapp [2] initiated the study of the parallel sum of nonlinear subdifferentials of convex functions in Hilbert spaces. Passty [11] considered the natural extension to nonlinear monotone operators. Recently, this kind of operations were investigated by Seeger [17], who introduced the notion of inverse addition of order p for pairs of convex sets.

The paper is organized as follows. The first primary goal is to give some algebraic properties between this two inverse operations. The intermediary purpose is to shed some light on the maximality of the inverse operations of two maximal monotone operators. In addition, we present here new conditions which assure the graph–convergence of this inverse operations. This permits us to extend some recent results of Attouch–Moudafi–Riahi [6] from Hilbert spaces to more general reflexive one. This appears to be interesting also from an applied viewpoint, by means of some examples.

2. PRELIMINARIES

In the sequel, we shall assume a certain familiarity with convex analysis for which we shall follow basically [5,8,19].

Throughout this note, we shall denote by X a real reflexive Banach space and X^* denotes its topological dual space. We shall denote strong and weak convergence by "$\xrightarrow{s}$" and "$\xrightarrow{w}$" respectively. The pairing between X and X^* will be denoted by $<\cdot,\cdot>$ and both the norms in X and X^* by $\|\cdot\|$ and $\|\cdot\|_*$.

If A is a subset of $X\times X^*$ and x is in X, we let $A(x)=\{y\in X^*;(x,y)\in A\}$, $D(A)=\{x\in X;A(x)\neq\emptyset\}$ its domaine and $R(A)=\cup\{A(x);x\in D(A)\}$ its range. The inverse mapping A^{-1} is defined by $A^{-1}(y)=\{x\in X;(x,y)\in A\}$. Obviously, we have $D(A^{-1})=R(A)$ and $(x,y)\in A$ iff $(y,x)\in A^{-1}$.

An operator A is called monotone if $<y_2-y_1,x_2-x_1>\geqslant 0$ whenever $(x_i,y_i)\in A$.

Such an operator is said to be maximal if it is not contained in any other monotone operator A' in $X\times X^*$.

As an example of maximal monotone operators, one can consider the subdifferential $A=\partial f$ of proper convex lower semicontinuous functions $f:X\longrightarrow \mathbb{R}\cup\{+\infty\}$ i.e., $\partial f(x)=\{y\in X^*;f(x)-f(u)\geqslant<y,x-u>\ \ \forall u\in X\}$. The same holds true, as it was introduced by Rockafellar, for the subdifferential of convex–concave saddle functions $F:X\times Y\longrightarrow \mathbb{R}\cup\{-\infty,+\infty\}$ i.e., $\partial F(x^*,y^*)=\partial F(\cdot,y^*)(x^*)\times\partial(-F(x^*,\cdot))(y^*)$.

DEFINITION 2.1. Let A,B be two operators in $\mathcal{M}(X)$. Let $\lambda\in\mathbb{R}$. The inverse addition between A and B is the operator

$$(A\oplus B)(x):=\cup\{A(x_1)\cap B(x_2);x=x_1+x_2\}$$

If we set $A_0(x)=X^*$ if $x=0$ and $A_0(x)=\emptyset$ otherwise, then $A_0\in\mathcal{M}(X)$ and the inverse scalar multiplication by λ is given by

$$\forall x\in X,\ (\lambda\circledcirc A)(x)=A(x/\lambda) \text{ if } \lambda\neq 0 \text{ and } (\lambda\circledcirc A)(x)=A_0(x) \text{ if } \lambda=0.$$

Summarizing the linear properties of this two inverse operations, we have the following.

THEOREM 2.2. *Let A,B and C be three operators in $\mathcal{M}(X)$. Let $\lambda,\mu\in\mathbb{R}_+$. Then we have:*

(1) $(\lambda\circledcirc A)\oplus(\mu\circledcirc B)\in\mathcal{M}(X)$.

(2) $A\oplus B=B\oplus A$ *and* $A_0\oplus B=B$.

(3) $(A\oplus B)\oplus C=A\oplus(B\oplus C)$ *and* $\lambda\circledcirc(\mu\circledcirc A)=(\lambda\cdot\mu)\circledcirc A$.

(4) $(A\oplus B)^{-1}=(A^{-1}+B^{-1})$ *and* $(\lambda\circledcirc A)^{-1}=\lambda A^{-1}$.

(5) $\lambda\circledcirc(A\oplus B)=(\lambda\circledcirc A)\oplus(\lambda\circledcirc B)$, $1\circledcirc A=A$ *and* $0\circledcirc A=A_0$.

(6) $\overbrace{A\oplus\cdots\oplus A}^{p\text{ times}}=p\circledcirc A$.

(7) $A_\lambda=(\lambda\circledcirc J)\oplus A$ *is exactly the well known Yosida approximation. Here* $J=\partial(\frac{1}{2}\|\cdot\|^2)$ *denotes the duality mapping of X.*

The (designation) of these operations have not been chosen arbitrarily as it could be seen in the property (4) above.

3. Maximality of the Inverse Operations

THEOREM 3.1. *Let A and B be two maximal monotone operators in $X\times X^*$. Let $\lambda,\mu\in\mathbb{R}_+$. Suppose that zero is an absorbing point of $R(A)-R(B)$:*

$$\bigcup_{\alpha>0}\alpha(R(A)-R(B))=X^*. \tag{1}$$

Then $(\lambda\circledcirc A)\oplus(\mu\circledcirc B)$ is a maximal monotone operator.

THEOREM 3.2. *Suppose that X is a Hilbert space. Then*

$$\bigcup_{\alpha>0}\alpha(R(A)-R(B))=Y \text{ is only a closed linear subspace of } X^* \tag{2}$$

implies that $(\lambda\circledcirc A)\oplus(\mu\circledcirc B)$ is maximal.

Remarks 2.3. 1) The conditions (1) and (2) in Theorem 3.1 and 3.2 are quite abstract, but it can be applied directly in many cases. For example if $0\in\operatorname{int}(R(A)-R(B))$ (the interior) or $0\in\operatorname{ri}(R(A)-R(B))$ (the relative interior), see [13].

2) If one of these two monotone operators is locally bounded, one deduces

(see [8,19]) that its range is the whole space X^*. Hence (1) is satisfied, and $(\lambda \odot A) \oplus (\mu \odot B)$ is a maximal.

3) We cannot delete the condition "zero is an absorbing point of $R(A)-R(B)$" in Theorem 3.1. For instance, if $X=\mathbb{R}^2$, A (resp., B) is the subdifferential of the support function on U (resp. V): $\sigma_U(x^*)=\sup\{<x^*,x>; x\in U\}$, $A=\partial\sigma_U$ and $B=\partial\sigma_V$ with $U=\{(u,v)\in X; u^2\leqslant v\}$ and $V=\mathbb{R}\times\{0\}$.

Then A and B are maximal monotone, but $A\oplus B$ is strictly included in the monotone operator $\{0\}\times\mathbb{R}^2$, since $(A^{-1}+B^{-1})(0)=\{0\}\times\mathbb{R}$. Hence $A\oplus B$ is not maximal. For the condition (1), $\cup_{\alpha\geqslant 0}\,\alpha(R(A)-R(B))=\cup_{\alpha\geqslant 0}\,\alpha(U-V)=\mathbb{R}\times\mathbb{R}_+$ is not a subspace of $\mathbb{R}^2$.

4) The result of Theorem 3.1 does not hold for an arbitrary Banach space, as the following example shows. If X is not reflexive, it follows from Jame's Theorem that there exists $y_0\in X^*$ such that the norm $\|y_0\|_*=\sup\{<y_0,x>; \|x\|\leqslant 1\}$ doesn't attein its supremum. Let $A=\partial(\|\cdot\|)$ and $B=\partial(<y_0,\cdot>^2)$. Then A and B are maximal monotone, $R(A)=B^*=\{y\in X^*; \|y\|_*\leqslant 1\}$ and $R(B)=\mathbb{R}y_0=\{\alpha y_0; \alpha\in\mathbb{R}\}$.

It follows that $\cup_{\alpha\geqslant 0}\,\alpha(R(A)-R(B))=X^*$. However the monotone operator $A\oplus B=(B^{-1}/B^*)^{-1}$ is strictly included in B. So that the assumption (1) is satisfied, but $A\oplus B$ is not maximal.

4. Graph Convergence of the Inverse Operations

Let us now examine how the perturbations of the above inverse operations can be stable. That's find a type for which when the monotone operators and scalars converge, the inverse sum and scalar product converge too. This provide flexible tools we give in what follow.

Definition 4.1. 1) Let (A_n) be a sequence of maximal monotone operators. We say that A_n graph–converge to an operator A iff for every $(x,y)\in A$, there exists $(x_n,y_n)\in A_n$ such that $x_n \xrightarrow{s} x$ and $y_n \xrightarrow{s} y$.

From monotonicity of the operators A_n, this graph–convergence can be formulated as the Kuratowski–Painlevé convergence of A_n to A.

2) (Uniform Brezis–Crandall–Pazy condition) Let X be a reflexive Banach space and X^* its dual space. For (A_n,B_n) a couple of maximal monotone operators in $X\times X^*$, let u_λ^n the unique solution of the inclusion

$$x^*\in Ju_\lambda^n + A_n u_\lambda^n + (B_n)_\lambda u_\lambda^n.$$

Indeed, this follows from the maximality of the operator $A_n+(B_n)_\lambda$ (see [19], Chap. 32).

We shall say that (A_n,B_n) satisfy the uniform Brezis–Crandall–Pazy condition (or uniforme BCP condition) iff for every $x^* \in X^*$

$$\limsup_{n\to+\infty} \alpha_n(x^*) < +\infty \quad \text{where} \quad \alpha_n(x^*) = \sup_{\lambda>0} \|(B_n)_\lambda(u_\lambda^n)\|.$$

We recommend [5,6,8,14,19] for more details concerning the main luggage and results used in this note.

Let us now consider some recent results, whose proofs are in [14, Chap. 7], which characterize the graph–convergence of the usual Minkowski sum of maximal operators:

THEOREM 4.2. *Consider (A_n) and (B_n) two sequences of maximal monotone operators which graph–converge to operators A and B. Then for every $\lambda>0$, $(A_n+(B_n)_\lambda)$ graph–converges to $(A+B_\lambda)$.*

THEOREM 4.3. *Consider (A_n) and (B_n) two sequences of maximal monotone operators in $X\times X^*$ which graph–converge to operators A and B. Suppose the uniform Brezis–Crandall–Pazy condition is satisfied by (A_n,B_n). Then (A_n+B_n) graph–converges to $(A+B)$.*

THEOREM 4.4. *Under the assumptions of Theorem 4.2. Suppose the uniform Brezis–Crandall–Pazy condition is satisfied by (A_n^{-1},B_n^{-1}).*

Then $(A_n \oplus B_n)$ graph–converges to $(A\oplus B)$.

We refer to [6] for the case of Hilbert spaces, when the graph–convergence is replaced with the bounded Hausdorff convergence.

5. APPLICATION

Various examples coming from optimization problem can be combined. In the end of this note we focuss our attention on approximate problems. Given A and B two operators, and $\epsilon>0$ a small parameter (intended to go to zero), we consider the following approximate problems:

$(\mathscr{P}_\epsilon^n)$ *find* $x_\epsilon^n \in X$ *such that* $(A_n\oplus B_n)(x_\epsilon^n)\cap B_\epsilon \neq \emptyset$.

$(\mathscr{P}_0)$ *find* $x_0\in X$ *such that* $0\in(A\oplus B)(x_0)$.

Let us consider x_n a solution to the problem $(\mathscr{P}_{\epsilon_n}^n)$, and let $y_n\in(A_n\oplus B_n)(x_n)$ such that $\|y_n\|\leqslant \epsilon_n$.

Suppose that the uniform BCP condition is satisfied by (A_n^{-1}, B_n^{-1}), that (A_n) and (B_n) graph–converge to A and B. Using Theorem 4.4 it follows that $(A_n \oplus B_n)$ graph–converge to $(A \oplus B)$.

Suppose now that the sequence (x_n) is bounded and that ϵ_n converges to zero. Then we can without loss of generality suppose that $x_n \xrightarrow{w} x_0$ and $y_n \xrightarrow{s} 0$. Passing to the limit we get $0 \in (A \oplus B)(x_0)$; which implies that x_0 is a solution to the problem $(\mathscr{P}_0)$.

One can summarize this property in the following Theorem.

THEOREM 5.1. *Under the assumptions of Theorem* 4.4. *Let us assume that for every integer n, there exists* $\epsilon_n > 0$ *and a solution* x_n *to* $(\mathscr{P}\epsilon_n)$. *Then every w–limit point* x_0 *of the sequence* (x_n), *let* $x_{\nu(n)} \xrightarrow{w} x_0$, *is a solution to the problem* $(\mathscr{P}_0)$.

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