Dowker-Type Theorems in Finite Dimensional Spaces

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INTRODUCTION

Let $C$ be a subset of a metric space. We shall say that $C$ is a convex body if it is a compact convex set with a non-empty interior.

In $\mathbb{R}^m$ with the norm induced by the usual scalar product, there are some estimators, not necessarily a metric, of "the distance" between convex bodies. In what follows $d$ shall be one of those estimator functions.

For $n = m + 1$, $m + 2$, ... let $\mathcal{P}_n$ be the set of all convex polytopes having at most $n$ vertices. Given a convex body $C$ we define $\mathcal{P}_n^i(C)$ to be the set of polytopes of $\mathcal{P}_n$ contained in $C$ and let similarly $\mathcal{P}_n^e(C)$ denote the set of polytopes of $\mathcal{P}_n$ containing $C$. We shall write $\mathcal{P}_n^i$, $\mathcal{P}_n^e$ instead of $\mathcal{P}_n^i(C)$, $\mathcal{P}_n^e(C)$.

Let $\delta : \mathbb{N} \rightarrow \mathbb{R}$ the function defined by $\delta(n) = \inf \{d(C, P) : P \in \mathcal{P}_n(C)\}$, where $C$ is a fixed convex body. In the same way shall be defined the functions $\delta^e(n)$ and $\delta^i(n)$, when $P \in \mathcal{P}_n^e(C)$, $\mathcal{P}_n^i(C)$ respectively. When referring to the three functions at one time, they will be called $\delta(n)$.

Given $C$ and $P$, two arbitrary convex bodies, let us define the functions:

$$\rho_1(C, P) = \sup_{z \in C} \inf_{y \in P} |z - y| \quad \text{and} \quad \rho_2(C, P) = \rho_1(P, C).$$

If it is clear which are the convex bodies we refer to, we shall denote these functions as $\rho_1$ and $\rho_2$.

Some classical theorems of Dowker [1] about packing and covering problems promoted the study of the convexity of this type of functions.

Eggleston in [2], while working on those topics, constructed a convex body $C$ in $\mathbb{R}^2$ for which the $\delta(n)$ functions, when $d(C, P) = \delta^e(C, P) = \rho_1 + \rho_2$, are not convex. On the other hand Gruber [3] leaves it open the question of whether or not the $\delta(n)$ functions are convex, when $d(C, P) = \delta^H(C, P) = \max \{\rho_1, \rho_2\}$.
the Hausdorff distance.

In this paper we answer this question negatively. We shall prove that for the Eggleston’s polygon the $\delta(n)$ functions are not convex. We will also construct in $\mathbb{R}^m$, $m \geq 2$, a convex body for which the $\delta(n)$ functions are not convex, when $d(C,P) = f(\rho_1, \rho_2)$, where $f$ is a function belonging to $\Omega = \{ f : \mathbb{R}^+ \times \mathbb{R}^+ \cup \{(0,0) \rightarrow \mathbb{R} : f(x,0) = x, f(0,y) = y, \max \{z,y\} \leq f(x,y) \}$.

The functions $f_\lambda(x,y) = (x^\lambda + y^\lambda)^{1/\lambda}$, $\lambda \in (0,\infty)$ and $f_\alpha(x,y) = \max \{z,y\}$ belong to $\Omega$.

RESULTS

The case $\mathbb{R}^2$

The convex body $C$ fixed in Eggleston’s example, is a regular $2r$ sided polygon of side-length $k$, and let $X_{2r}$ denote it. We shall see that for $X_{2r}$ the $\delta(n)$ functions are not convex when $f \in \Omega$.

**Lemma 1.** There exists $W \in \mathcal{P}_r^i$ and $Z \in \mathcal{P}_r^c$ such that

$$
\rho_1(X_{2r},W) = K \sin(\pi/2r),
$$

(1)

$$
\rho_2(X_{2r},Z) = \frac{k}{2} \tan(\pi/r).
$$

(2)

**Proof.** To prove (1) let $W$ be the polygon formed by joining $r$ alternate vertices of $X_{2r}$. For (2) we take $Z$, a polygon formed by producing $r$ alternate sides of $X_{2r}$. \hfill \Box

Now we take $r$ large enough to let us construct two polygons in which the following lemma is based on.

**Lemma 2.** For $r$ large enough, the following is true:

$$
\delta(2r-1) \geq K \sin(\pi/r) / \{4[1 + \cos^2(\pi/2r)]\} = \beta.
$$

(3)

**Proof.** We begin by constructing two regular polygons of $2r$ sides, $X'$ and $X''$ such that $X' \subset X_{2r} \subset X''$. If $\ell_i'$, $\ell_i$, $\ell_i''$ are the sides of $X'$, $X_{2r}$, $X''$ respectively, then the polygons will verify that, for each $i$, $\ell_i'$, $\ell_i$, $\ell_i''$ are parallel and $\beta$ denotes the distance between $\ell_i'$ and $\ell_i$, and between $\ell_i''$ and $\ell_i$.

Take a vertex $\ell_i''$ of $X''$ formed by the sides $\ell_i''$ and $\ell_i''_{i+1}$, and construct the triangle $C_i$ joining $\ell_i''$ to the mid-points of $\ell_i''$ and $\ell_i''_{i+1}$, which shall be denoted by $h_i$ and $h_{i+1}$ respectively. These triangles have disjoint interiors.
The value of $\beta$ has been chosen so that the polygon formed by joining $h_i$ on adjacent sides is also the polygon formed by the bisectors of the angles $\angle(L_i, L_i')$.

Let $L$ be a polygon such that $d(X_{2r}, L) < \beta$ and let $v_i$ be the vertex of $X_{2r}$ contained in the interior of $C_i$. The open ball $B(v_i, \beta) \subset \text{int}(C_i)$. From the definition of $d$, there exists $f \in \Omega$ such that $d(X_{2r}, L) = f(\rho_1(X_{2r}, L), \rho_2(X_{2r}, L))$ therefore $f(\rho_1(X_{2r}, L), \rho_2(X_{2r}, L)) < \beta$ from where

(a) $\rho_1(X_{2r}, L) < \beta \implies$ there exists $x \in L$ such that $x \in B(v_i, \beta) \subset \text{int}(C_i)$,
(b) $\rho_2(X_{2r}, L) < \beta$, therefore $L \subset \mathcal{X}$.

From (a) we conclude that the straight line joining $h_i$ and $h_{i+1}$ meets in $	ext{int}(L)$, and from (b) $L$ has at least one vertex in each $\text{int}(C_i)$. Therefore $L$ has at least $2r$ vertices. This proves boundary (3).

To finish the proof, one must bear in mind the inequalities $\delta(n) \leq \delta^i(n)$ and $\delta(n) \leq \delta^c(n)$, and suppose that $\delta(n)$ is a convex function. Then using the lemmas 1, 2 and considering that a convex function $g(n)$ of the integral variable $n$, such that $g(p) = 0$, verifies that $g(p-r) \geq r g(p-1)$ $(r = 1, \ldots, p-1)$, the following inequalities are true

$$K \sin(\pi/2r) = \rho_1(X_{2r}, W) \geq \delta^i(r) \geq \delta(r) \geq r \delta(2r-1) \geq$$
$$\geq r K \sin(\pi/r)/\{4[1 + \cos^2(\pi/2r)]\}. \tag{4}$$

But inequality (4) is false if $r$ is large. Similarly, it may be shown that $\delta^i(n)$ and $\delta^c(n)$ are not convex.

The case $\mathbb{R}^3$

We are going to construct in $\mathbb{R}^3$ a polytope with $4r$ vertices for which the $\delta(n)$ functions, when $d(C, P) = f(\rho_1, \rho_2)$, and $f \in \Omega$, are not convex. This polytope shall be denoted by $X_{4r}$ and will be constructed based on $X_{2r}$. Through induction on $m$, a convex body in $\mathbb{R}^m$, for which the $\delta(n)$ functions, when $d(C, P) = f(\rho_1, \rho_2)$ and $f \in \Omega$, are not convex.

**Definition 1.** We say that $A \subset \mathbb{R}^n$ is a *straight polytope* with $n-1$ dimensional bases and height $\xi$, if there exists a hyperplane $H$ in $\mathbb{R}^n$ which contains a polytope $B$ such that $A = \{x + \lambda u : x \in B \text{ and } |\lambda| \leq \xi\}$, where $u$ is a vector of norm 1, orthogonal to $H$.

We consider in $\mathbb{R}^3$ the $z = 0$ plane and $X_{2r}$ contained in it. We are going to work with the polytope $X_{4r} = \{z + \lambda u : z \in X_{2r} \text{ and } |\lambda| \leq \xi\}$, when $u = (0, 0, 1)$.
Lemma 3. There exists $W^* \in \mathcal{P}_{2\xi}$ and $Z^* \in \mathcal{P}_{2\xi}$ such that

$$\rho_1(X_{4r}, W^*) = K \sin(\pi/2r),$$

$$\rho_2(X_{4r}, Z^*) = \frac{K}{2} \tan(\pi/r).$$

Proof. To prove (5), construct a straight polytope $W^*$ with base $W$ (see (1)) and height $\xi$. To demonstrate (6), construct a different straight polytope $Z^*$ with base $Z$ (see (2)) and height $\xi$.

In the proof of the following inequality a family of convex bodies appears. From a certain large value of $\xi$, it is possible to affirm that each convex body contains a ball with centre at a vertex of $X_{4r}$ and radius $\beta$, so that the interiors of the convex bodies are disjoint.

Lemma 4.

$$\delta(4r - 1) \geq K \sin(\pi/r)/\{4[1 + \cos^2(\pi/2r)]\} = \beta.$$  

Proof. Construct the straight polytopes $X' = \{z + \lambda u : z \in X_r, |\lambda| \leq \xi - \beta\}$ and $X'' = \{z + \lambda u : z \in X''_r, |\lambda| \leq \xi + \beta\}$, where $X'_r$ and $X''_r$ are the $X'$ and $X''$ constructed in the proof of Lemma 3.

We shall construct a family of $4r$ convex bodies with disjoint interiors, and such that each open ball with centre at a vertex of $X_{4r}$ and radius $\beta$ is contained inside one of the convex bodies. Therefore, reasoning as in the proof of Lemma 2, any polytope $L$ such that $d(X_{4r}, L) < \beta$ would have at least $4r$ vertices. This proves (7).

The convex bodies are defined in the following way: let $C_i$ be the tetrahedron whose vertices are $u^i$, a vertex of $X''$, and the mid-points on $u^i$ edges. We can assert that $B(u^i, \beta) \subset \text{int}(C_i)$ if $\xi$ is large, and that the interiors of $C_i$ are disjoint.

For $n = 4r$, $\delta(n) = 0$, if $\delta(n)$ was convex, reasoning as we did at the end of the case $\mathbb{R}^2$, one would obtain the inequality

$$K \sin(\pi/2r) \geq 2r K \sin(\pi/r)/\{4[1 + \cos^2(\pi/2r)]\}$$

which is false for large $r$. Then $\delta(n)$ cannot be convex. Analogously, it can be shown that $\delta^i(n)$ and $\delta^e(n)$ are not convex.

Remarks. An example in $\mathbb{R}^m$ can be constructed by induction.
REFERENCES

