Modular Bernstein Algebras

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0. INTRODUCTION AND PRELIMINARIES

Let $A$ be an algebra. We will put $L(A)$ for its lattice of subalgebras, where $\leq$, $\land$, $\lor$ are naturally defined ($\leq$ is the inclusion relation). We define the length of $A$ (and put $l(A)$) as the length of $L(A)$. It is very easy to prove that if $A$ is a solvable algebra, then its length and its dimension coincide. In general we have that $l(A) \leq \dim_K A$ (if $K$ is the ground field). Throughout this paper we will deal with algebras over infinite fields of characteristic not two.

DEFINITION. An algebra $A$ is called modular (semimodular) if $L(A)$ is modular (semimodular).

DEFINITION. A Bernstein $K$-algebra $A$ is a commutative finite-dimensional algebra over $K$ for which there exists a nonzero algebra homomorphism $\omega: A \rightarrow K$ such that $(z^2)^2 = (\omega(z))^2 \cdot z^2$, for any $z$ in $A$.

The homomorphism $\omega$ is unique and is called the weight homomorphism of $A$. It is readily seen that, given $(A, \omega)$ a Bernstein algebra, for any $e$, a nonzero idempotent in $A$ (which obviously exists), there exists a Peirce decomposition of $A$: $A = K e \uplus \text{Ker} \omega$, $\text{Ker} \omega = U_e \uplus V_e$, where

$$U_e = \{ z \in A \mid ez = \frac{1}{2} z \} = \{ ze \mid e \in \text{Ker} \omega \}, \quad V_e = \{ z \in A \mid ez = 0 \} .$$

If $r = \dim_K U_e$, $s = \dim_K V_e$, the pair $(r + 1, s)$ is called the type of the Bernstein algebra $A$ and does not depend on the choice of $e$. It is also shown that $\dim_K(U_e)^2$ and $\dim_K(U_e V_e + (V_e)^2)$ do not depend on the choice of the idempotent.

DEFINITION. A Bernstein algebra $(A, \omega)$ is called exclusive if $(U_e)^2 = 0$ for

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some (equivalently for all) nonzero idempotent \( e \) in \( A \). A Bernstein algebra \((A, \omega)\) is said to be \textit{genetic} if \( \text{Ker} \omega \) is nilpotent.

\textbf{Theorem.} Let \( A \) be a Bernstein algebra. Then the following are equivalent:

(i) \( A \) is modular.

(ii) \( A \) is semimodular.

(iii) For any pair of subalgebras of \( A, S \) and \( T, S \oplus T \) is a subalgebra of \( A \).

\textbf{Proposition.} Let \( A \) be a trivial Bernstein algebra. Then \( A \) is modular if and only if \( A \) has type \((n+1,0)\) or \((1,n)\).

\textbf{Lemma.} A modular Bernstein algebra is always exclusive.

1. \textbf{Modular Bernstein Algebras over Arbitrary Fields}

Let \((A, \omega)\) be a non-trivial Bernstein algebra. If \( z \) is an element in \( \text{Ker} \omega \), we can consider \( m(z) \), the largest positive integer such that the elements \( z, z^2, \ldots, z^{m(z)} \) are linearly independent. Let us fix \( e \), a nonzero idempotent in \( A \). If we take \( 0 \neq v \) in \( V_e \), it can be seen that \( m(v) \geq 2 \) and moreover, from the definition of \( m(v) \), we get that the algebra generated by \( v \) is \( Kv \oplus Kv^2 \oplus \cdots \oplus Kv^{m(v)} \).

\textbf{Proposition.} Let \( A \) be a non-trivial modular Bernstein algebra of type \((r+1,s)\) and \( e \) a nonzero idempotent in \( A \). Then for all \( 0 \neq v \) in \( V_e \) we have:

\[ m(v) = r+1, \quad U_e = Kv^2 \oplus \cdots \oplus Kv^{r+1}. \]

\textbf{Theorem.} Let \( A \) be a non-trivial modular Bernstein algebra of type \((r+1,s)\) and \( e \) a nonzero idempotent in \( A \). Then, for any \( v \) in \( V_e \), \( v^{r+2} = 0 \).

\textbf{Theorem.} Let \((A, \omega)\) be a non-trivial Bernstein algebra of type \((r+1,s)\). Then the following are equivalent:

(i) \( A \) is a modular algebra.

(ii) \( A \) is exclusive and for all nonzero idempotent \( e \) in \( A \) and any \( 0 \neq v \) in \( V_e \):

\[ U_e = Kv^2 \oplus \cdots \oplus Kv^{r+1} \quad \text{and} \quad v^{r+2} = 0. \]

(iii) \( A \) is exclusive and there exists a nonzero idempotent \( e \) in \( A \) such that for any \( 0 \neq v \) in \( V_e \):

\[ U_e = Kv^2 \oplus \cdots \oplus Kv^{r+1} \quad \text{and} \quad v^{r+2} = 0. \]

(iv) \( A \) is exclusive and the subalgebras of \( A \) contained in \( \text{Ker} \omega \) are contained in \( U_e \) or contain \( U_e \).
\((v)\) A is exclusive and the subalgebras of \(A\) contained in \(\ker \omega\) are exactly the subspaces of \(U_e\) and the ideals of \(A\) contained in \(\ker \omega\).

2. MODULAR BERNSTEIN ALGEBRAS OVER ALGEBRAICALLY CLOSED FIELDS

The natural question in this moment is if all modular Bernstein algebras are necessarily genetic. First of all, we see that this question has a negative answer over arbitrary fields:

**Example.** Let \(K\) be \(\mathbb{Q}\) and \(A\) a commutative \(K\)-algebra of dimension 6, given by:

\[
A = Ke + Ku_1 + Ku_2 + Ku_3 + Kv + Kw, \quad \text{where} \quad e^2 = e, \quad eu_i = \frac{1}{2}u_i, \quad i = 1, 2, 3,
\]

\[
ev = ev = 0, \quad u_iu_j = 0, \quad i, j = 1, 2, 3, \quad u_1v = u_2, \quad u_2v = u_3, \quad u_3v = 0, \quad u_1w = 0,
\]

\[
u_2w = u_1, \quad u_3w = -u_2, \quad v^2 = u_1, \quad w^2 = u_3, \quad vw = \frac{3}{2}(-u_1 + u_3).
\]

Define the linear map \(\omega: A \rightarrow K\) by \(\omega(e) = 1, \quad \omega(u_i) = 0, \quad i = 1, 2, 3, \quad \omega(v) = \omega(w) = 0\). It is straightforward to see that \((A, \omega)\) is a Bernstein algebra. It can be seen that \((A, \omega)\) is modular. Nevertheless, \(A\) is not genetic since \(0 \neq u_1 = u_2w = (u_1v)w = ((u_2w)v)w = \ldots\).

**Theorem.** Any modular Bernstein algebra over an algebraically closed field \(K\) of characteristic different from two is genetic.

**Corollary.** Let \(K\) be an algebraically closed field of characteristic different from two and \(A\) a non-trivial Bernstein algebra of dimension \(n + 1\) over \(K\). Then \(A\) is modular if and only if \(A\) is isomorphic to the commutative algebra:

\[
Ke + Ku_1 + \ldots + Ku_{n-1} + Kv, \quad \text{where} \quad e^2 = e, \quad eu_i = \frac{1}{2}u_i, \quad 1 \leq i \leq n - 1, \quad ev = 0,
\]

\[
u_iu_j = 0, \quad 1 \leq i, j \leq n - 1, \quad u_1v = u_{i+1}, \quad 1 \leq i \leq n - 2, \quad u_{n-1}v = 0, \quad v^2 = u_1.
\]

References


