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Boyd Index and Nonlinear Volterra Equations

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In mathematical models of some physical phenomena a new class of nonlinear Volterra equations appears [5],[6]. The equations belonging to this class have $u=0$ as a solution (trivial solution), but with respect to their physical meaning, nonnegative nontrivial solutions are of prime importance.

Consider the Volterra equation

$$u(x)=\int_0^x k(x-s)\,g(u(s))\,\mathrm{d}s,\quad x\in[0,1] \qquad (*)$$

where we assume that k satisfies the following conditions:

(k) $k\colon[0,1]\longrightarrow[0,+\infty)$ is an increasing absolutely continuous nonnegative function.

The restriction that, for physical considerations, we put on g are:

(g) $g\colon[0,+\infty)\longrightarrow[0,+\infty)$ is an increasing continuous function such that $g(0)=0$, $g(x)/x$ is decreasing and $g(x)/x\longrightarrow+\infty$ when $x\longrightarrow 0^+$.

In previous papers (see [6]) necessary and sufficient conditions for the existence of nontrivial solutions of these equations were given. They can be considered as extensions of classical Grinpenberg's condition [5], and they are formulated in terms of integrals of functions involving the data.

On the other hand, their form suggest a similarity with some integrals related to Boyd indices [1]. Boyd indices were introduced in connection with rearrengement invariant spaces and interpolation theory. However, regarding applications to nonlinear integral equations, the usual framework of Boyd indices is too narrow. This we demonstrate: notice that, due to standard extension theorems [2], we only need to know the form of the functions close to zero. So we are interested only in their lower Boyd index β_f^-. Nevertheless the following formula for a suitable function f with $\beta_{\bar{f}}>0$ holds: $\beta_{\bar{f}^{-1}}=1/\alpha_{\bar{f}}$. This undesirable behaviour is due to the passage from f to $\bar{f}$, which may strongly modify the function.

In our paper, we show a new connection between Boyd indices and nonlinear Volterra equations. What we do is to define some extensions of the notion of lower Boyd index and to show how, in combination with some extensions of Gripenberg's condition, can be used to determine whether the equations considered have or have not non trivial solutions.

DEFINITION. A continuous function $f\colon (0,\delta)\longrightarrow(0,+\infty)$ will be considered to belong to class C if the following limit, which we shall refer to as the Boyd index of f, exists

$$\beta_f=\lim_{x\to 0^+}\log f(x)/\log\tfrac{1}{x}.$$

METHOD FOR NATURAL KERNELS $k(t)=t^{\alpha}$, $\alpha\geqslant 1$

We present our method to decide whether the nonlinear Volterra equation

$$u(x)=\int_0^x (x-s)^{\alpha-1}g(u(s))\,\mathrm{d}s \qquad (**)$$

has or has not a nontrivial solution.

Due to physical considerations we assume that g satisfies conditions:

(g) g is a monotone increasing function, with $g(0)=0$, and such that $g(x)/x$ is monotone decreasing.

THEOREM. *Let g be a function of class C satisfying* (g), *and let $\alpha>1$. If $-1<\beta_g$ the equation*

$$u(x)=\int_0^x (x-s)^{\alpha-1}g(u(s))\,\mathrm{d}s$$

has a nontrivial solution.

EXAMPLE. Let $P(x_0,\dots,x_n)$ be a function of the form

$$P(x_0,\dots,x_n)=\textstyle\sum_{|\alpha|<M} a_i x_i^{\alpha},$$

where $x=(x_{i_0},\dots,x_{i_k})$, $i=(i_0,\dots,i_k)$, $i_k\in\{0,\dots,n\}$, $a_i\in\mathbb{R}$, $\alpha=(\alpha_0,\dots,\alpha_k)$, $\alpha_i\geqslant 0$, $\alpha_i\in\mathbb{R}$, and $|\alpha|=\max\alpha_i$. Let $g(t)=P(x_0(t),x_1(t),\dots,x_n(t))$, where $x_0(t)=t$ and $x_k(t)=\log\dots\log 1/t$ (k times), defined on a suitable interval $(0,\epsilon)$. If we assume $\partial x_0\leqslant 1$ to get g to satisfy (g) then the method applies in a very simple form, and completely determines, after n steps, the existence of nontrivial solutions.

METHOD FOR NON–VERY–SMOOTH KERNELS

THEOREM. *Let k be a kernel of class C satisfying* (k) *and g a function of class C satisfying* (g). *If $-\infty<\beta_k<0$ and $-1<\beta_g$ then equation* (*) *has a nontrivial solution.*

COUNTEREXAMPLE. The equation

$$u(x)=\int_0^x k(x-s)[u(s)]^p \, ds, p\in(0,1) \qquad (***)$$

where $k=K'$ and $K(x)=\exp(-\exp x^{-\alpha})$, $\alpha\geqslant 1$, has no nontrivial solution.

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