EXTRACTA MATHEMATICAE Vol. 6, Núms. 2 y 3, 166 – 169 (1991)

An Approach to Schreier's Space

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AMS *Subject Class.* (1980): 46B20, 46B15 *Received January* 31, 1992

In 1930, J. Schreier [10] introduced the notion of admissibility in order to show that the now called weak–Banach–Saks property does not hold in every Banach space. A variation of this idea produced the Schreier's space (see [1],[2]). This is the space obtained by completion of the space of finite sequences with respect to the following norm:

$$\|x\|_S = \sup_{A \text{ admissible}} \sum_{j\in A} |x_j| ,$$

where a finite sub–set of natural numbers $A=\{n_1<\cdots<n_k\}$ is said to be admissible if $k\leqslant n_1$.

In this extract we collect the basic properties of S, which can be considered mainly folklore, and show how this space can be used to provide counter examples to the three–space problem for several properties such as: Dunford–Pettis and Hereditary Dunford–Pettis, weak p–Banach–Saks, and S_p.

It can be easily verified that S is algebraically contained in c_0 and contains ℓ_1. No other space ℓ_p is algebraically contained in S: consider the sequence

$$u=(1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\overbrace{\tfrac{1}{8},\cdots\cdots,\tfrac{1}{8}}^{8\text{ times}},\overbrace{\tfrac{1}{16},\cdots\cdots\cdots,\tfrac{1}{16}}^{16\text{ times}},\cdots)$$

which belongs to ℓ_p for all $p>1$, and has norm 1 in S^{**}. However, a suitable "right shift" of u originates a sequence not in S^{**}:

$$u_1,0,\cdots,0,u_2,u_3,0,\cdots,0,u_4,u_5,u_6,0,\cdots,0,u_7,u_8,u_9,u_{10},0,\cdots$$

where u_2 is in the place 2+4, u_4 is in the place 8+16+32, etc.

It is routine to verify that the canonical vectors (e_i) form an unconditional basis for S. Moreover:

(1) S is a subspace of a certain $C(K)$ space with K countable.

To show this, let us consider the product space $\{-1,0,1\}^{\mathbb{N}}$, and define $K=$

$\{(\epsilon_n)\in\{-1,0,1\}^{\mathbb{N}} / \text{support of } (\epsilon_n) \text{ is admissible}\}$. K is closed, and hence compact, and countable. The application $j: S \longrightarrow C(K)$ carrying $x=(x_n)$ to the continuous function $j(x)$, defined by $j(x)((\epsilon_n)_{n\in A}) = \Sigma_{n\in A} x_n \epsilon_n$ is an isometry:

$$\|j(x)\|_\infty = \sup_{(\epsilon_n)\in K} |j(x)(\epsilon_n)| = \sup_{A \text{ admissible}} \sum_{n\in A} |x_n| = \|x\|_S$$

From (1), it immediately follows:

(2) A sequence (x^n) of S is weakly convergent to x if and only if, for every j, the sequence of j^{th} coordinates $(x_j^n)_{n\in\mathbb{N}}$ converges to x_j.

(3) S^* is separable (since the unit ball of S is metrizable in the weak topology).

The space S contains isometric copies of c_0: the sequence

$$s_n = 2^{-n+1}(e_{2^{n-1}} + \cdots + e_{2^n-1})$$

spans in S an isometric copy of c_0. In fact, it follows from (1) that any infinite–dimensional closed subspace of S contains an isomorphic copy of c_0 (see also [2, Prop. 2.10]). The fact of being "hereditarily c_0" prevents S from having subspaces isomorphic to ℓ_p for any $1 \leqslant p < +\infty$. The space S provides another example to show that being "hereditarily c_0" in the sense that "any closed infinite–dimensional subspace contains a subspace isomorphic to c_0", and in the sense that "any normalized weakly null sequence admits a sub–sequence equivalent to the canonical basis of c_0" are different properties: S contains weakly null sequences, such as the canonical basis (e_n) of S having no subsequence equivalent to the canonical basis of c_0, since no sub–sequence (e_{i_m}) of it satisfies for some constant K an estimate of the form

$$\sup_N \left\| \sum_{m=1}^{N} e_{i_m} \right\| \leqslant K.$$

MAIN RESULTS

A Banach space X is said to have the Dunford–Pettis property (DPP) if any weakly compact operator $T: X \longrightarrow Y$ transforms weakly compact sets of X into relatively compact sets of Y. Equivalently, given weakly null sequences (x_n) and (x_n^*) in X and X^* respectively, $\lim \langle x_n^*, x_n \rangle = 0$. L_1 and $C(K)$ spaces are examples of spaces with DPP. A Banach space X is said to have the hereditary Dunford–Pettis property ($\mathrm{DPP_h}$) if any closed subspace of X has the DPP. ℓ_1

and c_0 are examples of spaces having the DPP_h. A deep characterization, due to Elton (see [5, Cor. 3.5]), of this property is: any normalized weakly null sequence admits a sub-sequence equivalent to the canonical basis of c_0. Since the sequence (e_n) does not admit sub-sequences equivalent to the canonical basis of c_0, S does not have the hereditary Dunford–Pettis property, that is, it contains a subspace without the Dunford–Pettis property. Moreover:

(4) S does not have the Dunford–Pettis property

In fact, the unit vector sequence is weakly null in S^*: this immediately follows from the estimate

$$\left\| \sum_{k=1}^{2^N-1} e_{i_k} \right\|_{S^*} \leqslant N$$

(5) The natural inclusions $\ell_1 \longrightarrow S \longrightarrow c_0$ are weakly compact operators.

It is not hard to check that $S^{**} \subseteq c_0$, from which (5) follows. Since the set $\{e_n\}$ is not relatively compact in c_0, here we have an equivalent proof that S does not have the Dunford–Pettis property. On the other hand, given a bounded sequence in ℓ_1 one can easily extract a sub-sequence pointwise convergent to a certain sequence of ℓ_1.

A property P is said to be a three–space property if, whenever a closed subspace Y of a Banach space X and the corresponding quotient X/Y have P, then X also has P. For instance, it is easy to see that reflexivity or the Schur property are three–spaces. A problem which has been around for some years is whether the Dunford–Pettis prperty is a three–space property (see [4] and [7] for additional information). In [3] we solved this question in the negative by showing that $\ell_1 \oplus S$ contains a subspace H having the hereditary Dunford–Pettis property, such as $\ell_1 \oplus S/H \approx c_0$. Therefore the Dunford–Pettis and the hereditary Dunford–Pettis properties are not three–space.

That example also provides negative answers to questions raised in [6]: the weak p–Banach–Saks property and the S_p property (see below for definitions) are not three–space properties.

THEOREM. *The Dunford–Pettis, hereditary Dunford–Pettis, weak–p–Banach–Saks and S_p are not three–space properties.*

A sequence (x_n) in a Banach space X is said to be p–Banach–Saks, $p>1$, if

$$\|\sum_{k=1}^{n} x_k\| \leqslant C \cdot n^{1/p} \quad \text{for } p>1$$

for some constant $C>0$ and all $n \in \mathbb{N}$. It is said to be Banach–Saks if it has norm convergent arithmetic means. A Banach space is said to have the (weak) $p-$ Banach–Saks property when each (weakly null) bounded sequence (x_m) admits a sub–sequence (x_n) and a point x such that (x_n-x) is a p–Banach–Saks sequence. In [6] it is proved that the p–Banach–Saks and weak–p–Banach–Saks properties are "almost" three–space: if Y and X/Y have it, then X has, for each $\epsilon>0$, the $(p-\epsilon)$–Banach–Saks property. It is also proved that the Banach–Saks property is three–space. In [9] it is shown that the weak–Banach–Saks property is not three space. Closely related properties are the following: a sequence (x_n) in a Banach space is said to be weakly–p–summable ($p \geqslant 1$) if there is a $C>0$ such that

$$\sup_n \|\sum_{k=1}^{n} \xi_k x_k\| \leqslant C \|(\xi_n)\|_{\ell_{p^*}}$$

for any $(\xi_n) \in \ell_{p^*}$. We shall say that the sequence (x_n) is weakly–p–convergent to $x \in X$ if the sequence (x_n-x) is weakly–p–summable. A Banach space is said to have property W_p if any bounded sequence admits a weakly–p–convergent sub–sequence. The weak version of W_p property has been called ([8]) S_p property.

Problem. A problem which still remains open is: are p–Banach–Saks and W_p three–space properties?

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