On Some Generalizations of the Kakutani–Stone and Stone–Weierstrass Theorems

M. I. GARRIDO AND F. MONTALVO

Dpto. de Matemáticas, Univ. Extremadura, 06071 Badajoz, Spain

For a completely regular space X, $C^*(X)$ denotes the algebra of all bounded real−valued continuous functions over X. We consider the topology of uniform convergence over $C^*(X)$.

When K is a compact space, the Stone−Weierstrass and Kakutani−Stone theorems provide necessary and sufficient conditions under which a function $f \in C^*(K)$ can be uniformly approximated by members of an algebra, lattice or vector lattice of $C^*(K)$. In this way, the uniform closure and, in particular, the uniform density of algebras and lattices of $C^*(K)$, can be characterized. Other authors, like Császár, Nobeling, Bauer,..., have studied the uniform density of lattices of $C^*(K)$ with some additional properties (subtractive lattices, affine lattices, semi−affine lattices,...).

When X is a non−compact space, different versions and generalizations of those theorems have been given. Thus, Hewitt [7] gives a density uniform theorem for algebras of $C^*(X)$ containing all the real constant functions. On the other hand, Blasco [1] characterizes the uniform closure for certain lattices of $C^*(X)$.

In this paper we make a systematic study about uniform approximation for algebras and lattices of $C^*(X)$. If $\mathfrak{F}$ is an algebra or lattice (vector lattice, affine lattice,...) we shall characterize its uniform closure and we shall give necessary and sufficient conditions for uniform density in $C^*(X)$. For our purposes we shall identify the rings $C^*(X)$ and $C(\beta X)$ (βX is the Stone−Cech compactification of X). In this way, we generalize the classic results in the compact case. Likewise we also obtain, in particular, the generalizations by Hewitt and Blasco.

1. NOTATION

The space $C^*(X)$ can also be provided with an algebraic structure and an order structure, defining pointwise the suitable operations. Thus, if $\mathfrak{F}$ is a subset

of $C^*(X)$ we shall say that $\mathfrak{F}$ is:

1. a ring, if $f, g \in \mathfrak{F}$ then $f-g \in \mathfrak{F}$ and $fg \in \mathfrak{F}$.

2. a linear space, if $f, g \in \mathfrak{F}$ and $\lambda, \mu \in \mathbb{R}$ then $\lambda f + \mu g \in \mathfrak{F}$.

3. an algebra, if $\mathfrak{F}$ is a ring and a linear space.

4. a lattice, if $f, g \in \mathfrak{F}$ then $f \vee g \in \mathfrak{F}$ and $f \wedge g \in \mathfrak{F}$, where $f \vee g = \sup(f, g)$ and $f \wedge g = \inf(f, g)$

5. a vector lattice or linear lattice, if $\mathfrak{F}$ is a lattice and it is a linear space.

6. a subtractive lattice, if $\mathfrak{F}$ is a lattice and if $f, g \in \mathfrak{F}$ then $f-g \in \mathfrak{F}$.

7. an affine lattice, if $\mathfrak{F}$ is a lattice and if $f \in \mathfrak{F}$ and $a \in \mathbb{R}$ then $f+a \in \mathfrak{F}$ and $af \in \mathfrak{F}$.

8. a semi–affine lattice, if $\mathfrak{F}$ is a lattice and if $f \in \mathfrak{F}$ then $f + a \in \mathfrak{F}$ and $\mu f \in \mathfrak{F}$ for every $a \in \mathbb{R}$ and $\mu \in \Gamma$ where Γ is a set of real numbers containing 0 and unbounded both from above and from below.

Given $f \in C^*(X)$ and a real number a, we let $L_a(f) = \{x \in X / f(x) \leqslant a\}$, $L^a(f) = \{x \in X / f(x) \geqslant a\}$ and we refer to $L_a(f)$ and to $L^a(f)$ as the Lebesgue sets of f. We let $Z(f)$ denote the set $\{x \in X / f(x) = 0\}$ and call it zero–set of f.

Suppose that $\mathfrak{F} \subset C^*(X)$ and that A and B are subsets of X, we say that:

1. $\mathfrak{F}$ S_1–separates or completely separates A and B when there is $g \in \mathfrak{F}$ such that $0 \leqslant g \leqslant 1$, $g(x) = 0$ if $x \in A$ and $g(x) = 1$ if $x \in B$.

2. $\mathfrak{F}$ S_2–separates or completely separates A and B when there is $g \in \mathfrak{F}$ such that $\overline{g(A)} \cap \overline{g(B)} = \emptyset$.

3. $\mathfrak{F}$ S_i–separates ($i=1,2$) the Lebesgue sets of a function f when for every $a < b$, $\mathfrak{F}$ S_i–separates $L_a(f)$ and $L^a(f)$.

2. UNIFORM APPROXIMATION FOR SUBLATTICES OF $C^*(X)$

In this section we generalize the theorems by Kakutani, Stone, Csázsár,... for sublattices of $C^*(X)$. For this we identify $C^*(X)$ and $C(\beta X)$. This identification allows us to translate conditions related to points in βX into conditions related to Lebesgue sets or zero–sets in X.

THEOREM 1. *Let $\mathfrak{F}$ be a sublattice of $C^*(X)$ and let $f \in C^*(X)$. Suppose that for every $a < b$ and $\epsilon > 0$ there exists $g \in \mathfrak{F}$ such that $|g(x) - a| < \epsilon$ if $x \in L_a(f)$ and $|g(x) - b| < \epsilon$ if $x \in L^b(f)$. Then $f \in \overline{\mathfrak{F}}$.*

THEOREM 2. *Let $\mathfrak{F}$ be a sublattice of $C^*(X)$. The following conditions are equivalent:*

a) *$\mathfrak{F}$ is uniformly dense in $C^*(X)$.*

b) *For every pair of disjoints zero$-$sets in X, Z_1 and Z_2, for every pair of real numbers a, b and $\epsilon > 0$ there exists $g \in \mathfrak{F}$ such that $|g(x) - a| < \epsilon$ if $x \in Z_1$ and $|g(x) - b| < \epsilon$ if $x \in Z_2$.*

Remark. The sufficient condition in Theorem 1 is also necessary for sublattices containing all the real constant functions.

THEOREM 3. *Let $\mathfrak{F}$ be a semi$-$affine sublattice of $C^*(X)$ and let $f \in C^*(X)$. The following conditions are equivalent:*

a) *$f \in \overline{\mathfrak{F}}$.*

b) *For every $a < b$ and $\epsilon > 0$ there exists $g \in \mathfrak{F}$ such that $|g(x) - a| < \epsilon$ if $x \in L_a(f)$ and $|g(x) - b| < \epsilon$ if $x \in L^b(f)$.*

c) *For every $a < b$ there exists $g \in \mathfrak{F}$ such that $g(x) = a$ if $x \in L_a(f)$ and $g(x) = b$ if $x \in L^b(f)$.*

d) *$\mathfrak{F}$ $S_1$$-$separates the Lebesgue sets of f.*

e) *For every $a < b$ there exists $g \in \mathfrak{F}$ such that $\sup\{g(x) : x \in L_a(f)\} < \inf\{g(x) : x \in L^b(f)\}$, whenever $L_a(f)$ and $L^b(f)$ are not empty.*

f) *$\mathfrak{F}$ separates the Lebesgue sets of f.*

Remarks. 1) In [1], Blasco has been pointed out the equivalence between the conditions *a)* and *e)*.

2) It is not difficult to find some examples showing that, in order that a function is in the uniform closure of a sublattice containing all the real constant functions, the conditions *e)* and *f)* are only necessary, the condition *c)* is only sufficient and the condition *d)* is neither necessary nor sufficient.

3) Since every affine, subtractive or vector sublattice which contains all the real constant functions is a semi$-$affine sublattice, then the above result remain true for them.

THEOREM 4. *If, in Theorem 3, we change "Lebesgue sets of a function f" by "every pair of disjoints zero$-$sets in X", we obtain the uniform density of $\mathfrak{F}$.*

3. UNIFORM APPROXIMATION FOR SUBALGEBRAS OF $C^*(X)$

As before, we shall now make a similar study for subalgebras of $C^*(X)$. In this way we shall generalize the Stone$-$Weierstrass theorem and we shall obtain,

as a consequence, the well–known results by Hewitt.

THEOREM 5. *Let $\mathfrak{F}$ be a subalgebra of $C^*(X)$ and let $f \in C^*(X)$. Then $f \in \overline{\mathfrak{F}}$ if and only if the following conditions are satisfied:*

  i) *$\mathfrak{F}$ separates the Lebesgue sets of f.*

  ii) *For every $\epsilon > 0$ there are $\delta > 0$ and $g \in \mathfrak{F}$ such that $L^\epsilon(|f|) \subset L^\delta(g)$.*

THEOREM 6. *Let $\mathfrak{F}$ be a subalgebra of $C^*(X)$. Then $\mathfrak{F}$ is uniformly dense in $C^*(X)$ if and only if the following conditions are satisfied:*

  i) *$\mathfrak{F}$ separates every pair of disjoints zero–sets in X.*

  ii) *$\mathfrak{F}$ contains a unity of $C^*(X)$ (i.e., there is $f \in \mathfrak{F}$ with $f \geq \epsilon > 0$).*

Remark. In [6], Hewitt shows that any subalgebra of $C^*(X)$ which separates every pair of disjoints zero–sets in X and containing all the real constant functions is uniformly dense in $C^*(X)$. But, note that this result is now a corollary of the above theorem.

REFERENCES

1. J.L. BLASCO, Hausdorff compactifications and Lebesgue sets, *Topology and its Applications* 15 (1983), 111–117.
2. A. CSÁSZÁR, On approximation theorems for uniform spaces, *Acta Math. Acad. Sci. Hungaricae* 22 (1971), 177–186.
3. A. CSÁSZÁR AND J. CZIPSZER, Sur des criteries generaux d'approximation uniforme, *Ann. Univ. Sci. Budapest* 6 (1963), 17–26.
4. M.I. GARRIDO, "Aproximación Uniforme en Espacios de Funciones Continuas", Ph. D. Thesis, Publ. Dpto. Mat. UNEX, Vol. 24, Badajoz, 1990.
5. M.I. GARRIDO, On some generalizations of the Kakutani–Stone and Stone Weierstras theorem, to appear in *Acta Math. Hungarica.*
6. L. GILLMAN AND M. JERISON, "Rings of Continuous Functions", Springer–Verlag, New York, 1976.
7. E. HEWITT, Certain generalizations of the Weierstrass approximation theorem, *Duke Math.* 14 (1947), 419–427.
8. S. KAKUTANI, Concrete representation of abstract (M)–spaces, *Annals of Math.* 42(4) (1941), 994–1024.
9. S. MROWKA, On some approximation theorems, *Nieuw Archief voor Wiskunde* XVI(3) (1968), 94–111.
10. L. NACHBIN, "Elements of Approximation Theory", Van Nostrand Math. Studies, Princeton, 1967.
11. G. NOBELING AND H. BAUER, Allgemeine approximationskriterien mit anwendungen, *Jber. Deustch. Math. Verein* 58 (1955), 54–72.
12. M.H. STONE, Applications of the theory of Boolean rings to general topology, *Trans. Amer. Math. Soc.* 41 (1937), 375–481.
13. M.H. STONE, A generalized Weierstarss approximation theorem, *Math. Magazine* 21 (1948), 167–184, 237–254.