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Operational Quantities Derived from the Norm and Generalized Fredholm Theory [1]

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Several operational quantities, defined in terms of the norm and the class of finite dimensional Banach spaces, have been used to characterize the classes of upper and lower semi–Fredholm operators, strictly singular and strictly cosingular operators, and to derive some perturbation results.

In this paper we shall introduce and study some operational quantities derived from the norm and associated to a space ideal. By means of these quantities we construct a generalized Fredholm theory in which the space ideal plays the role of the finite dimensional spaces, and we extend to this situation many of the results of the classical Fredholm theory.

We use the following notation: X, Y are Banach spaces; X^* the dual space of X; I_X the identity operator of X; $L(X,Y)$ the space of the (linear and continuous) operators between X and Y; $n(T) := \|T\|$, $N(T)$ and $R(T)$, the norm, the kernel and the range of the operator $T \in L(X,Y)$, respectively $J_M : M \longrightarrow X$ the inclusion of the (closed) subspace M of X in X. Recall that T is upper semi–Fredholm ($T \in SF_+$) if $\dim(N(T)) < \infty$ and $R(T)$ is closed; T is strictly singular ($T \in SS$) if TJ_M isomorphism implies $\dim(M) < \infty$.

THE CLASSICAL CASE. Let $\mathbb{F}$ denote the class of all finite dimensional Banach spaces. For $X \notin \mathbb{F}$, let $S_{\mathbb{F}}(X)$ denote the set of subspaces $M \subset X$ such that $M \notin \mathbb{F}$; i.e., the infinite dimensional subspaces. If $T \in L(X,Y)$, we can derive from the norm the following operational quantities:

$$in_{\mathbb{F}}(T) := \inf\{ n(TJ_M) : M \in S_{\mathbb{F}}(X) \}$$
$$sin_{\mathbb{F}}(T) := \sup\{ in_{\mathbb{F}}(TJ_M) : M \in S_{\mathbb{F}}(X) \}.$$

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We have (see [2]):

$$T \in SF_+(X,Y) \iff in_{\mathbb{F}}(T) > 0$$
$$T \in SS(X,Y) \iff sin_{\mathbb{F}}(T) = 0$$
$$T \in SF_+ \text{ and } sin_{\mathbb{F}}(S) = 0 \implies T+S \in SF_+$$

GENERALIZED FREDHOLM THEORY. Recall that a space ideal $\mathbb{A}$ is a class of Banach spaces bigger than $\mathbb{F}$, which is stable under isomorphisms, finite products and complemented subspaces. Let $S_{\mathbb{A}}(X)$ denote the set of subspaces $M \subset X$ such that $M \notin \mathbb{A}$; and $\mathbb{A}_d := \{X : S_{\mathbb{A}}(X) \neq \emptyset\}$.

The $\mathbb{A}$–operational quantities are defined as follows. For $X \notin \mathbb{A}_d$ and $T \in L(X,Y)$,

$$in_{\mathbb{A}}(T) := \inf\{n(TJ_M) : M \in S_{\mathbb{A}}(X)\}$$
$$sin_{\mathbb{A}}(T) := \sup\{in_{\mathbb{A}}(TJ_M) : M \in S_{\mathbb{A}}(X)\}.$$

Now we introduce the class $S\mathbb{A}_+$ of generalized upper semi–Fredholm operators and the corresponding perturbation class $\mathbb{A}SS$ which extends the strictly singular operators:

$$S\mathbb{A}_+(X,Y) := L(X,Y) \text{ if } X \in \mathbb{A}_d\,; \quad := \{T \in L(X,Y) : in_{\mathbb{A}}(T) > 0\}, \text{ if } X \notin \mathbb{A}_d$$
$$\mathbb{A}SS(X,Y) := L(X,Y) \text{ if } X \in \mathbb{A}_d\,; \quad := \{T \in L(X,Y) : sin_{\mathbb{A}}(T) = 0\}, \text{ if } X \notin \mathbb{A}_d$$

Note that $SF_+ = S\mathbb{F}_+$ and $\mathbb{F}SS = SS$. Howewer $S\mathbb{A}_+$ is empty in some cases. We shall need the following notion [1]: X and Y are totally incomparable if Banach spaces isomorphic to a subspace of X and to a subspace of Y have finite dimension. Given a space ideal $\mathbb{A}$ we shall denote by $\mathbb{A}_i$ the class of Banach spaces which are totally incomparable with every space in $\mathbb{A}$. Note that $\mathbb{F} = \mathbb{F}_{ii}$ and if $\mathbb{A} = \mathbb{A}_{ii}$, $X \notin \mathbb{A}_d$ and $Y \in \mathbb{A}_d$, then $S\mathbb{A}_+(X,Y) = \emptyset$.

The most important properties of $S\mathbb{A}_+$ and $\mathbb{A}SS$ are the following:

$S\mathbb{A}_+$ is invariant by $\mathbb{A}SS$: $T \in S\mathbb{A}_+$ and $S \in \mathbb{A}SS \implies T+S \in S\mathbb{A}_+$.

$SF_+ \subset S\mathbb{A}_+$.

$\mathbb{A} = \mathbb{A}_{ii} \implies SS \subset \mathbb{A}SS$.

$S\mathbb{A}_+(X,Y)$ is open in $L(X,Y)$.

$\mathbb{A}SS(X,Y)$ is closed in $L(X,Y)$.

$T \in S\mathbb{A}_+(X,Y) \implies N(T) \in \mathbb{A}$.

$I_X \in \mathbb{A}SS \iff X \in \mathbb{A}_d$.

$\mathbb{A} = \mathbb{A}_{ii} \implies \mathbb{A}SS(X,Y) = \{T \in L(X,Y) : TJ_M \text{ isomorphism} \implies M \in \mathbb{A}_d\}$.

$\mathbb{A}=\mathbb{A}_{ii} \implies \mathbb{A}SS$ is an operator ideal.
$ST \in S\mathbb{A}_+ \implies T \in S\mathbb{A}_+$.

We do not know if the following implication is true: $S, T \in S\mathbb{A}_+ \implies ST \in S\mathbb{A}_+$; that is, if the class $S\mathbb{A}_+$ is a semigroup.

EXAMPLE. For certain ideals $\mathbb{A}$ (for example the reflexive Banach spaces, the spaces containing no copies of ℓ_1, the separable spaces, the superreflexive spaces, the B-convex spaces, the quasi-reflexive spaces), we obtain $R(T)$ closed and $N(T) \in \mathbb{A} \implies T \in S\mathbb{A}_+$.

Howewer, for the class of all Banach spaces such that every infinite dimensional subspace contains a copy of ℓ_2, $\mathbb{A}=(\ell_2)_{ii}$, the above result is not true: there exists a Banach space X containing a subspace U isometric to ℓ_2 such that the associated quotient map $X \longrightarrow X/U$ does not belong to $S\mathbb{A}_+$.

The class $S\mathbb{R}_+$, where $\mathbb{R}$ is the class of all reflexive spaces, is properly contained in the class of tauberian operators. Recall that $T \in L(X,Y)$ is tauberian when $T^{**-1} J_Y(Y) = J_X(X)$, where J_X is the canonical inclusion of X in X^{**}.

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