Quasi–Frobenius Quotient Rings

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AMS Subject Class. (1980): 16A08, 16A30, 16A60

Received March 6, 1991

Let \( R \) be an associative (not necessarily commutative) ring with unit. The study of flat left \( R \)-modules permits to achieve homological characterizations for some kind of rings (regular Von Neumann, hereditary). Colby investigated in [1] the rings with the property that every left \( R \)-module is embedded in a flat left \( R \)-module and called them left IF rings. These rings include regular and quasi–Frobenius rings. Another useful tool for the study of non–commutative rings is the classical localization, when it is possible, or the localizations constructed from the most general perspective of torsion theories (mainly, the maximal quotient ring). In a recent paper [3] we try to find a relation between these two approximations to the problem of the determination of the structure for general rings. The suggesting idea is that for commutative domains the class of torsionfree modules is exactly the class of submodules of flat modules.

If \( \mathcal{F}_0 \) denotes the class of left \( R \)-modules that embed in flat left \( R \)-modules, we investigate the rings for which this class is the torsionfree class for some hereditary torsion theory \( \tau_0 \) on the category of left \( R \)-modules, \( R\text{-Mod} \). These are the left \( FTF \) ("flat are torsionfree") rings. Analogously, we define right \( FTF \) rings (with notation \( \mathcal{F}'_0 \) and \( \tau'_0 \)). If \( R \) is a left \( FTF \) ring, the filter \( \mathcal{A}(\tau_0) \) of left ideal associated with \( \tau_0 \) has an easy description: A left ideal \( I \) of \( R \) is an element of \( \mathcal{A}(\tau_0) \) if and only if \( I \) contains a finitely generated left ideal \( I_0 \) such that \( \text{Hom}_R(R/I_0,R) = 0 \) [3, Prop. 4.5]. In this note we expose the results obtained in our research on \( FTF \) rings. The keys for the proofs of our results have been the description of the filter \( \mathcal{A}(\tau_0) \) and the well known result due to Lazard that asserts that the flat left \( R \)-modules are the direct limits of projective modules. For the undefined concepts the reader is referred to [7]. The following is the main result proved in [3].

**Theorem 1.** The following conditions are equivalents for a ring \( R 

i) \( R \) is a left \( FTF \) ring and satisfies D.C.C. on left annihilators.

ii) \( R \) has a twosided maximal quotient ring \( Q \) such that \( Q \) is a semiprimary left and right QP–3.

iii) \( R \) is a right \( FTF \) ring and satisfies D.C.C. on right annihilators.

In case i)–iii) hold, \( R \) is a \( \tau_0 \) and \( \tau'_0 \)-artinian.

Masake [5, Theorem 2] shows that rings satifying ii) are the rings \( R \) for which every

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1 This work has been partially supported by NATO and the grant PS88–108 from DGICYT.
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finitely generated $E(\tau R)$—torsionless left $R$—module is embedded in a free left $R$—module and $R$ satisfies D.C.C. on left annihilators. We will prove that for a ring with the property that every finitely generated $E(\tau R)$—torsionless left $R$—module embeds in a free left $R$—module, A.C.C. on left annihilators implies D.C.C. on left annihilators. To prove this statement we need to answer to the following question: When is a left FTF ring right FTF? For the definition of coherence relative to a torsion theory, we refer to [4].

PROPOSITION 2. If $R$ is a ring then $R$ is left FTF and $\tau_0$—coherent if and only if $R$ is right FTF and $\tau_0'$—coherent.

Morita proved [6] that a left noetherian ring with $E(\tau R)$ flat has a flat right injective hull too. As a corollary to the proof of the foregoing theorem we improve this result:

COROLLARY 3. If $R$ is left coherent and $E(\tau R)$ is flat then $R$ is a (left and right) FTF ring.

The following two results improve Masaike’s Theorem before cited.

PROPOSITION 4. Let $R$ be a ring with the property that every finitely generated $E(\tau R)$—torsionless left $R$—module embeds in a free left $R$—module. If $R$ satisfies A.C.C. on left annihilators then $R$ satisfies D.C.C. on left annihilators.

THEOREM 5. For a ring $R$ the following statements are equivalent:

i) Every finitely generated $E(\tau R)$—torsionless left $R$—module embeds in a free left $R$—module and $R$ satisfies A.C.C. on left annihilators.

ii) Every finitely generated $E(\tau R)$—torsionless left $R$—module embeds in a free left $R$—module and $R$ satisfies D.C.C. on left annihilators.

iii) $R$ is left (or right) FTF and $\tau_0$—(or $\tau_0'$—) noetherian.

iv) $R$ is left and right FTF and both $\tau_0$—artinian as $\tau_0'$—artinian.

v) $R$ has a twosided maximal quotient ring $Q$ such that $Q$ is semiprimary left and right QF—3.

The last results of this note give characterizations of the existence of a quasi—Frobenius left maximal (classical) quotient ring for some rings. These characterizations applies for left QF—3 rings (or more generally, for left QF—3' rings) and for left FPF rings with finite left Goldie dimension [2, Corollary 2.9].

THEOREM 6. For any ring $R$ the following statements are equivalent:

i) $R$ is a $\tau_0$—noetherian left FTF ring and $\tau_0$ is an exact torsion theory.

ii) $R$ has a twosided maximal quotient ring $Q$ such that $Q$ is quasi—Frobenius.

THEOREM 7. Let $R$ be a ring such that every finitely generated submodule of $E(\tau R)$ is torsionless. Then $R$ has a quasi—Frobenius left classical quotient ring if and only if

1) $R$ satisfies A.C.C. on right annihilators.

2) If $I$ is a finitely generated left ideal of $R$ such that $r(I) = 0$, then $I$ contains a regular element.
REFERENCES