ON THE $\lambda$–PROPERTY AND SPACES OF CONVERGENT SEQUENCES

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For $X$ a normed space, we will use the following notation

$$B(X) = \{ x \in X : \|x\| \leq 1 \}$$

$$\mbox{ext}(B(X)) = \{ e \in X : e \mbox{ is an extreme point of } B(X) \}.$$ 

We will denote by $c(X)$ (resp. $l_1(X)$) the space of convergent (resp. absolutely summmable) sequences of elements of $X$ with its usual norm.

The following concepts were introduced by Aron and Lohman in [2]:

Let $x$ be an element of $B(X)$. If there exist $e \in \mbox{ext}(B(X))$, $y \in B(X)$ and $\lambda \in [0,1]$ such that $x = \lambda e + (1-\lambda)y$, we will say that the triple $(e,y,\lambda)$ is amenable to $x$. In this case we can define

$$\lambda(x) = \mbox{Sup} \{ \lambda \in [0,1] : \mbox{there exist } (e,y,\lambda) \mbox{ amenable to } x \}.$$ 

$X$ is said to have the $\lambda$–property if each point in its unit ball admits an amenable triple, and it is defined the $\lambda$–function of $X$ as the function $x \mapsto \lambda(x)$ from $B(X)$ into $[0,1]$.

It is said that $X$ has the uniform $\lambda$–property if $X$ has the $\lambda$–property and, in addition, satisfies

$$\inf \{ \lambda(x) : x \in B(X) \} > 0.$$ 

It can be easily checked that every strictly convex space has the uniform $\lambda$–property.

We are interested in the $\lambda$–property for spaces of the form $c(X)$. Concerning this, up to date, it is only known that, if $X$ is a strictly convex space, $c(X)$ has the uniform $\lambda$–property and for each $x = \{x_n\} \in B(c(X))$ we have

$$\bar{x}(x) = \inf \{ \lambda(x_n) : n \in \mathbb{N} \}.$$ 

where $\bar{x}(\cdot)$ (resp. $\lambda(\cdot)$) denotes the $\lambda$–function of $c(X)$ (resp. $X$),
we will use the same notation throughout the paper. This was proved by Aron – Lohman [2] and Alipruru [1].

It should be also mentioned that Aron – Lohman proved in [2] that every finite-dimensional normed space has the uniform \( \lambda \)-property.

In order to study the \( \lambda \)-property for \( c(X) \) we need the following characterization of the extreme points of its unit ball:

If \( X \) is a normed space and \( e = \{ e_n \} \in B(c(X)) \) then
\[
e \in \text{ext}(B(c(X))) \quad \text{if, and only if,} \quad e_n \in \text{ext}(B(X)), \forall \ n \in \mathbb{N}.
\]
Using this characterization, it is easy to prove the next result.

**Proposition.** Let \( X \) be a normed space. If \( c(X) \) has the \( \lambda \)-property (resp. uniform \( \lambda \)-property), then \( X \) has the \( \lambda \)-property (resp. uniform \( \lambda \)-property).

The converse of the above result is not true as the following example shows:

**Example.** Let \( C' \) denote the convex hull of the union of the sets \( A_1 \) and \( A_2 \) given by
\[
A_1 = \{ (x,y,z) \in \mathbb{R}^3 : |x|, |y| \leq 1 , z = 0 \}
\]
\[
A_2 = \{ (x,y,z) \in \mathbb{R}^3 : x^2 + z^2 = 1 , y = 0 , z \geq 0 \}
\]

take \( C = (0,0,1) + C' \) and \( \| - \| \) the norm on \( \mathbb{R}^3 \) whose unit ball is \( B = \text{co}(C \cup (-C)) \).

Then \( X = (\mathbb{R}^3, \| - \|) \) has the uniform \( \lambda \)-property and \( c(X) \) fails to have the \( \lambda \)-property.

Remark: The above space \( X \) appears in [2].

As we have said before, up to date, it has been proved that \( c(X) \) has the \( \lambda \)-property only when \( X \) is a strictly convex space.

The next result gives a class of non strictly convex normed spaces \( X \) for which \( c(X) \) has the uniform \( \lambda \)-property.

**Theorem.** Let \( X \) be a finite-dimensional normed space whose unit ball is a polyhedron (\( \text{ext}(B(X)) \) is a finite set), then \( c(X) \) has the uniform \( \lambda \)-property and, for each \( x = \{ x_n \} \in B(c(X)) \)
\[
\hat{x}(x) = \inf \{ \lambda(x_n) : n \in \mathbb{N} \}.
\]
In the above theorem $X$ has the uniform $\lambda$-property. Now we see that this is not necessary in order to get that $c(X)$ has the $\lambda$-property. That will be a consequence of the following result.

**THEOREM.** Let $X$ be a normed space. The space $c(X)$ has the $\lambda$-property if, and only if, $c(l_1(X))$ has the $\lambda$-property.

**COROLLARY.** Let $X$ be a normed space satisfying one of the two conditions:

a) $X$ is strictly convex

b) $B(X)$ is a polyhedron,

then $c(l_1(X))$ has the $\lambda$-property.

Taking into account that spaces of the form $l_1(X)$ always fail to have the uniform $\lambda$-property [3], the comment before the theorems is now clear.

**REFERENCES**

[1] Alzpuru Tomás, A., Una extensión del teorema de Tietze y la $\lambda$-propiedad en $C(K,X)$, XIV Jornadas Hispano–Lusas de Matemáticas, 1.989.

