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MODULI OF ROTUNDITY AND SMOOTHNESS FOR CONVEX BODIES

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1. Introduction and Notation

Let X denote a real (finite- or infinite-dimensional) normed linear space. A subset K of X is a *body* if K is closed, bounded, convex and has nonempty interior. When 0 (the origin of X) is an interior point of K, K^π denotes the polar body of K, i.e. $K^\pi=\{ f \in X^*: f(x) \le 1 \ \forall x \in K \}$. A *rooted body* is an ordered pair (K,r) where K is a body and r, the root, is an interior point of K.

In [3], in studying the possibility to tile Banach spaces by means of bodies, three properties of a rooted body (K,r) have been considered, properties expressed in terms of finite-dimensional sections or projections of K through r, which came out to be natural extensions, from balls to bodies, of classical concepts as uniform rotundity, smoothness and nonsquareness. In [4] very simple analytic reformulations of these properties are given. In particular

- A body K, with boundary ∂K and norm-diameter d_K, is *uniformly convex* (*UC*) if for any $\varepsilon \in (0,d_K)$ there exists a positive $\delta = \delta(\varepsilon)$ such that $\mathrm{dist}\left(\frac{x+y}{2},\partial K \right) \ge \delta$ whenever x,y are points in K with $\|x-y\| \ge \varepsilon$.
- A rooted body (K,r) is *uniformly smooth* (*US*) if K is a smooth body and the duality mapping $J : \partial K \to S^*$ (the unit sphere of the dual space X^*) is norm to norm uniformly continuous (for each $x \in \partial K$, J_x is the only element in S^* such that the hyperplane $J_x^{-1}(\lambda)$ supports K-r at x-r for some $\lambda > 0$).
- A body K is *US* if (K,r) is US for some (hence for any) interior point r of K.

Of course, the property of being UC or US for a body is invariant under renorming. A very interesting fact is that the classical duality relationship between uniform rotundity and uniform smoothness is still valid, in the sense that, if the origin is an interior point of K, K is UC iff K^π is US. Moreover, if the Banach space X admits a body that is UC or US, X has to be reflexive (in fact, super-reflexive), that forces all bodies in X to be weakly compact.

This extension of uniform properties to bodies suggests the possibility to extend the notions of moduli of rotundity and smoothness too. Here we present a few steps we have made in this direction.

2. Moduli of Rotundity and Smoothness

There are many choices one can make in defining the modulus of rotundity for a body K, actually they are many even when K is a ball (see, for instance, [1]). The choice one makes obviously depends on the properties one wishes to preserve in passing from balls to bodies: the well known characterization of rotundity (X is a rotund space iff $\delta(2) = 1$) seemed to us the most natural property to be preserved. In this context, the first remark is that, in absence of any kind of simmetry in the generic body, we need to use something more strictly related to the body than the norm. The idea is to consider a rooted body and to use its Minkowski functional with respect to the root, since it describes the body itself in the best way. The second modification to the classical definition is realized, following an idea of Gurarii ([2]), by replacing the middle point of the segment $[x,y]$ with one of the points of this segment that are closest to the root. At our best knowledge, it is still an open problem whether the classical modulus and the Gurarii's one coincide, even for the balls.

For a body K that has 0 as an interior point, let q_K denote its Minkowski functional, i.e., for $x \in X$, set $q_K(x) = \mathit{inf}\ \{ \alpha > 0 : x \in \alpha K \}$. For a rooted body (K,r) let $s(K,r)$ denote its Minkowski diameter (M-diameter), i.e. set $s(K,r) = \mathit{sup}\ \{ q_{K-r}(x-y) : x,y \in K \}$. Of course, $s(K,r) > 2$ whenever K is not centrally symmetric with respect to r. Let $\Delta_{(K,r)} : [0,s(K,r)) \to [0,1]$ be defined as follows

$$\Delta_{(K,r)} = \mathit{inf}\ \{ \mathit{max}\ \{ 1-q_{K-r}(tx+(1-t)y) : 0 \le t \le 1 \} : x,y \in K-r,\ q_{K-r}(x-y) \ge \varepsilon \}.$$

If we denote by $\delta_{(K,r)}$ our proposed modulus of rotundity, for $0 \le \varepsilon < s(K,r)$ we can set $\delta_{(K,r)}(\varepsilon) = \Delta_{(K,r)}(\varepsilon)$. Now the problem is the definition of $\delta_{(K,r)}$ at $s(K,r)$. This is the step in which the situation of the generic body seems to be very different from the situation of the ball. In fact we can prove, for example, that, for $1 \le p < \infty$, there exists in ℓ^p a rooted body whose M-diameter is not attained; we conjecture that such a body exists in any infinite-dimensional Banach space. In this situation we consider two cases.

If $\dim X = 2$ set $\delta_{(K,r)}(s(K,r)) = \Delta_{(K,r)}(s(K,r))$.

If $\dim X > 2$ set

$$\delta_{(K,r)}(s(K,r)) = \mathit{inf}\ \{ \Delta_{((K-r)\cap F,0)}(s((K-r)\cap F,0)) : F \text{ two dimensional subspace of } X \}.$$

It is obvious that $\delta_{(K,r)}$ is nondecreasing on $[0,s(K,r))$ (on $[0,s(K,r)]$ if $\dim X = 2$). Easy examples show that, if K is not rotund, it may happen that $\delta_{(K,r)}$ is not monotone (and continuous) at $s(K,r)$. Actually the most important properties of $\delta_{(K,r)}$ we know can be summarized as follows:

- K is UC iff for any $r \in \mathrm{int}K$ we have $\delta_{(K,r)}(\varepsilon) > 0$ whenever $\varepsilon > 0$;
- if K is rotund, then for any $r \in \mathrm{int}K$ we have $\delta_{(K,r)}(s(K,r)) = 1$;
- if K is not rotund, then there exists $r \in \mathrm{int}K$ such that $\delta_{(K,r)}(s(K,r)) < 1$.

We conclude by showing that things are much simpler for the modulus of smoothness. For notational simplicity assume $r = 0$. The modulus of smoothness $\rho_{(K,0)} : [0,\infty) \to [0,\infty)$ can be defined as follows

$$\rho_{(K,0)}(\tau) = sup\left\{q_K\left(\frac{x+y}{2}\right)+q_K\left(\frac{x-y}{2}\right)-1 : x \in \partial K,\ q_K(y) = \tau\right\}.$$

It turns out that

- $\rho_{(K,0)}$ is continuous and convex on $[0,\infty)$;
- K is US iff q_K is K-uniformly Frechet differentiable;
- K is US iff $\lim_{\tau\to 0+}\tau^{-1}\rho_{(K,0)}(\tau) = 0$.

Moreover, the classical Lindenstrauss' duality relationships between the modulus of smoothness and the modulus of rotundity actually become inequalities, this being due uniquely to the fact of considering the so called "Gurarii's version" of the modulus of rotundity in place of the "middle point version". More precisely we have for any positive τ

$$sup\ \{\ \tau\varepsilon/2 - \delta_{(K,0)}(\varepsilon) : 0 \le \varepsilon < s(K,0)\ \} \le \rho_{(K,0)}(\tau) \le$$
$$\le sup\ \{\ (\tau\varepsilon - \delta_{(K,0)}(\varepsilon))/2 : 0 \le \varepsilon < s(K,0)\ \}$$

and similar inequalities hold when K^{π} and K are interchanged.

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