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ON BANACH SPACES WHICH ARE M-IDEALS IN THEIR BIDUALS.

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INTRODUCTION

A Banach space X is an M-ideal in its bidual if the relation

$$\|f+w\| = \|f\| + \|w\|$$

holds for every f in X^* and every w in $X^\perp$.

The class of the Banach spaces which are M-ideals in their biduals, in short, the class of M-embedded spaces, has been carefull investigated, in particular by A. Lima, G. Godefroy and the "West Berlin School". The spaces $c_0(I)$ $-I$ any set- equipped with their canonical norm belong to this class, which also contains e.g. certain spaces $K(E,F)$ of compact operators between reflexive spaces (see [7]). This class has very nice properties; for instant, these are Weakly Compactly Generated (W.C.G.) Asplund spaces [2; Th. 3], have the property (v) [5; Th. 1] and (u) [4; Main Th.] of Pelczynski and satisfy , among other isometric properties, that "every isometric isomorphism of X^{**} is the bitranspose of an isometric isomorphism of X" [6; Prop. 4.2]. The purpose of this work is to show that these properties are also trues in a more wide class of Banach spaces. In short, we prove

THEOREM

Let X be a nonreflexive Banach space and $0<\varepsilon\leq 1$. If X the relation

$$\|f+g\|\geq\varepsilon\|f\|+\|g\|$$

holds for every f in X^* and every g in $X^\perp$ (the annihilator of X), then:

i) X is an Asplund space.

ii) X has the property (u)of Pelczynski.

iii) Every isometric isomorphism of X^{**} is the bitranspose of an isometric isomorphism of X.

iv) X admits a shrinking projectional resolution of identity.

v) No proper subspace of X^* is norming of X.

The proof of the theorem require, on the one hand, a most carefull reading of the proof of the announced results for the class of the M-embedded spaces. In this way, for instance, the assertions *ii)* and *iv)* are direct consequence of this analysis and the subsequent improvements, such as the nice stability behaviour and the proof of the following lemma.

LEMMA 1

Let X and ε as above and F in X^{**}. Then F/B_{X^*} is the difference of *two bounded lower semicontinuous functions on* (B_{X^*}, w^*).

On the other hand, *i)* is a trivial consequence of a "old" result of Giles, Gregory and Sims [3, th. 3.3], while the assertion *iii)* is a trivial consequence of the following result, which is an interesting improvement of [6; Prop. 4.2]

LEMMA 2.

Let X *and* ε *be as above and a Banach space* Y *such that its natural projection* π_Y $(\pi_Y(Y^{***}) = Y^*$ *and* Ker $\pi_Y = Y^{\perp})$ *is bicontractive* $(\|\pi_Y\| = 1 = \|1 - \pi_Y\|)$. *Then, every isometric linear mapping from* X^{**} *onto* Y^{**} *is the bitranspose of an isometric linear mapping from* X *onto* Y.

Finally, for the assertion *v)*, we will use the following results:

LEMMA 3.[G. Godefroy]

Let X *be a Banach space . Then the following assertions are equivalent:*

a) $\bigcap_{x \in X} B_{X^{**}}(x, \|F - x\|) = \{F\}$ *for all* F *in* $X^{**} \setminus X$.

b) No proper subspace of X^* *is norming of* X.

LEMMA 4.

Let X *and* ε *be as above and* F *in* $X^{**} \setminus X$. *Then* $\bigcap_{x \in X} B_{X^{**}}(x, \|F - x\|) = \{F\}$.

The interest of the theorem lies in its numerous and interesting consequences. In the next result, we collect some of them, precisely, those which are known for the M-embedded spaces.

COROLLARY.

Let X *and* ε *as above. Then:*

1) X *contains no an isomorphic copy of* ℓ_1.

2) $\pi_{X^{**}}$ *is the only one-projection from* $X^{(4)}$ *onto* X^{**}.

3) π_{X^*} *is the only one-projection* P *in* X^{***} *with* $KerP = X^{\perp}$.

4) X^* *is strong unique predual of* X^{**} *(for every Banach space* Y, *every isometric isomorphism from* X^* *onto* Y^* *is* w^**-continuous).*

5) X *has a shrinking M-basis.*

6) X *is W.C.G.*

7) There are operators $T: X \longrightarrow c_0(I)$ *and* $S: X^* \longrightarrow c_0(I)$, *card* I = *dens* X *such that* T^{tt} *is injective and* S *is* w^*-w^**-continuous and injective.*

8) X *has an equivalent LUR-norm whose dual norm is also LUR. In particular his norm is F-differentiale and LUR.*

9) X *has the property (v) of Pelczynski.*

10) c_0 *is complemented in* X.

11) X^* *contains a complemented subspace isomorphic to* ℓ_1.

12)X^* *is weakly sequentially complete.*

13) Every operator from X *to a space not containing* c_0 *(in particular: every operator from* X *to* X^**) is wealky compact.*

14) X *does not weakly sequentially complete.*

15) X *does not have the Radon Nikodym property.*

16) X is not complemented in X^{**}.
17) X^{**}/X is not separable.
18) If $X \subseteq Y \subseteq X^{**}$ and q is an one-projection from Y onto X, then X coincides with Y.
19) Every subspace or quotient of X which is isomorphic to a dual space is reflexive.
20) $Z(X) \cong Z(X^{**})$, where by Z(X) we denote the centralizer of X (for definition , we refer to [1; p. 53 ff]).

EXAMPLE .

Finally, we show an example of a Banach space X that holds, for ever $f \in X^*$ and $g \in X^{\perp}$, the relation

$$\|f+g\| \geq (1/2)\|f\| + \|g\|,$$

which is not M-ideal in its bidual: Let X denotes the space c_o renormed by

$$|(a_n)| = Max \{ Sup\{|a_{2n}|;\ n \in \mathbb{N}\},\ Sup\{|a_{2n-1}|;\ n \in \mathbb{N}\} + (1/4)Sup\{|a_{2n}|;\ n \in \mathbb{N}\},\ (3/2)Sup\{|a_{2n-1}|;\ n \in \mathbb{N}\}\}.$$

REFERENCES

[1] E. BEHRENDS: *M-structure and the Banach-Stone-Theorem.* Lecture Notes in Math., Vol. 736. springer-Verlag, Berlin (1979)

[2] M. FABIAN and G. GODEFROY: The dual of every Asplund space admits a projectional resolution of the identity. *Studia Math. XCI.* (1988), 141-151.

[3] J.R. GILES, D.A. GREGORY and B. SIMS: Geometrical implications of upper semi-continuity of the duality mapping on a Banach space. *Pacif. J. Math.* Vol. 79 N1 (1978), 99-109.

[4] G. GODEFROY and D.LI: Banach spaces which are M-ideals in their bidual have property (u). *Ann. Inst. Fourier, Grenoble* 39, 2 (1989), 361-371.

[5] G. GODEFROY and P. SAAB: Weakly unconditionally convergent series in M-ideals. *Math. Scand.* 64 (1989), 307318.

[6] P. HARMAND and A. LIMA: Banach spaces which are M-ideals in their biduals. *Trans. Amer. Math. Soc.* 283 (1984), 253-264.

[7] A. LIMA: M-ideals of compact operators in classical Banach spaces. Math. Scand. 44 (1979), 207-217.