EXTRACTA MATHEMATICAE 5, n.2, 71-73 (1990)

ON OPERATOR IDEALS DETERMINED BY SEQUENCES

Manuel González, *University of Cantabria, 39005 Santander, Spain*
Antonio Martinón, *University of La Laguna,38271 La Laguna (Tenerife), Spain*

(Dedicated to the memory of José M. García Lafuente)

Tacon [TA], by using techniques of non-standard analysis, showed that if $T: X \longrightarrow X$ is a (linear and continuous) operator on the Banach space X, then T^m is compact for some m if and only if for every bounded sequence $(x_n) \subset X$ there exists p such that $(T^p x_n)$ is a relatively compact sequence. This result is obtained as a corollary of the following theorem: if $T_n: X \longrightarrow Y$ is a sequence of operators mapping a Banach space X into a Banach space Y, and for every bounded sequence $(x_n) \subset X$ there exists p such that $(T_p x_n)$ is relatively compact, then T_m is compact for some m. Similar results were showed for weakly compact operators.

Using standard techniques Barría [BA] proved the above results in the case $X=Y$ a Hilbert space; Brown-Foias [BF] in the case $X=Y$ a Banach space. Moreover Buoni-Klein-Scott-Wadhwa [BK1,2] proved the results of Tacon, and analogous results for other classes of operators (completely continuous, weakly completely continuous and Rosenthal operators).

In this paper, using the $\mathcal{A}$-variation associated to an operator ideal $\mathcal{A}$, due to Astala [AS], we introduce a family of operators ideals for which, using analogous techniques to that in [BF], we prove in a unified way the results of Tacon. This family includes all the previously considered classes [TA, BA, BF, BK1, BK2], and some others.

In the following X, Y, Z will be Banach spaces; $\mathcal{L}(X,Y)$ the class of all operators between X and Y; B_X the closed unit ball of X; $\ell_\infty(X)$ the space of all bounded sequences (x_n) in X attached with the sup-norm. The range of the sequence (x_n) will be denoted by $\{x_n\}$.

Supported in part by DGICYT grant PB88-0417.

THE MAIN RESULT. Let $\mathcal{A}$ be an operator ideal in the sense of Pietsch [PI], and let D be a bounded subset of X. Then the $\mathcal{A}$-variation $h_{\mathcal{A}}(D)$ of D is defined [AS] as follows

$$h_{\mathcal{A}}(D) := \inf \{\varepsilon>0 : \exists Z,\ \exists K\in\mathcal{A}(Z,X),\ D\subset KB_Z+\varepsilon B_X\} .$$

Using $h_{\mathcal{A}}$ we can define the following subspaces of $\ell_\infty(X)$ [GM1, GM2]

$$s_{\mathcal{A}}(X) := \{(x_n)\in\ell_\infty(X) : h_{\mathcal{A}}(\{x_n\})=0\} .$$

Let $\mathcal{A}$, $\mathcal{B}$ be operator ideals. We define the operator ideal $(\mathcal{A},\mathcal{B})$ in the following way:

$$(\mathcal{A},\mathcal{B})\ (X,Y) := \{T\in\mathcal{L}(X,Y) : (x_n)\in s_{\mathcal{A}}(X) \Rightarrow (Tx_n)\in s_{\mathcal{B}}(Y)\} .$$

1 Theorem. Let $\mathcal{A}$, $\mathcal{B}$ be operator ideals. For a sequence of operators $(T_n)\subset\mathcal{L}(X,Y)$, the following assertions are equivalent:

(1) For every $(x_n)\in s_{\mathcal{A}}(X)$, there is $p\in\mathbb{N}$ such that $(T_p x_n)\in s_{\mathcal{B}}(Y)$

(2) There exists $m\in\mathbb{N}$ such that $T_m\in(\mathcal{A},\mathcal{B})$

2 Corollary. Let $\mathcal{A}$, $\mathcal{B}$ be operator ideals, and $T\in\mathcal{L}(X,X)$. There exists $m\in\mathbb{N}$ such that $T^m\in(\mathcal{A},\mathcal{B})$ if and only if for every $(x_n)\in s_{\mathcal{A}}(X)$, there is $p\in\mathbb{N}$ such that $(T^p x_n)\in s_{\mathcal{B}}(X)$

APPLICATIONS. Note that the class $\mathcal{L}$ of all operators is an operator ideal such that $h_{\mathcal{L}}=0$ and $s_{\mathcal{L}}=\ell_\infty$.

3 Example. The compact operators $\mathcal{C}o$ **:**

$$T\in\mathcal{C}o \iff ((x_n) \text{ bounded} \Rightarrow \{Tx_n\} \text{ relatively compact}) .$$

We have that $\mathcal{C}o = (\mathcal{L},\mathcal{C}o)$.

4 Example. The weakly compact operators $\mathcal{WC}o$ **:**

$$T\in\mathcal{WC}o \iff ((x_n) \text{ bounded} \Rightarrow \{Tx_n\} \text{ weakly relatively compact}) .$$

We have that $\mathcal{WC}o = (\mathcal{L},\mathcal{WC}o)$.

5 Example. The Rosenthal operators $\mathcal{R}o$ **:**

$$T\in\mathcal{R}o \iff ((x_n) \text{ bounded} \Rightarrow (Tx_n) \text{ has a weakly Cauchy subsequence})$$

We have that $\mathcal{R}o=(\mathcal{L},\mathcal{R}o)$.

6 Example. The completely continuous operators $\mathcal{CC}$ **:**

$$T\in\mathcal{CC} \iff ((x_n) \text{ weakly convergent} \Rightarrow (Tx_n) \text{ convergent})$$

We have that $\mathcal{CC}=(\mathcal{WC}o,\mathcal{C}o)=(\mathcal{R}o,\mathcal{C}o)$.

7 Example. The weakly completely continuous operators $\mathcal{WCC}$ **:**

$$T\in\mathcal{WCC} \Leftrightarrow ((x_n) \text{ weakly Cauchy} \Rightarrow (Tx_n) \text{ weakly convergent})$$

We have that $\mathcal{WCC}=(\mathcal{WC}a,\mathcal{R}a)$.

8 Example. The V* operators V^* **:**

$$T\in V^* \Leftrightarrow (\text{for every } S\in\mathcal{L}(Y,\ell_1),\ ST\in\mathcal{C}a)$$

We have that $V^*=(\mathcal{L},V^*)$

9 Example. The strictly cosingular operators $\mathcal{SC}$:

$$T\in\mathcal{SC}(X,Y) \Leftrightarrow (\text{for every quotient } Y/U,\ Q_U T \text{ surjection} \Rightarrow dim(Y/U)<\infty)$$

where $Q_U : Y \longrightarrow Y/U$ is the quotient map. Clearly $(\mathcal{L},\mathcal{SC})$ contains $\mathcal{SC}$, and it is an open problem if both operator ideals coincide. However we can show that the equality it is equivalent to an old open problem:

10 Proposition $\mathcal{SC}=(\mathcal{L},\mathcal{SC})$ if and only if every infinite dimensional Banach space has an infinite dimensional separable quotient.

11 Proposition. If every infinite dimensional quotient of Y (of X) has a infinite dimensional separable quotient, then $(\mathcal{L},\mathcal{SC})(X,Y)=\mathcal{SC}(X,Y)$ for every X (for every Y).

REFERENCES

[AS] Astala: *On measures of noncompactness and ideal variations in Banach spaces.* Ann. Acad. ci. Fenn. Ser. A. I. Math. Diss. 29 (1980).

[BA] Barría: *On power compact operators.* Proc. Amer. Math. Soc. 80 (1980), 123-124.

[BF] Brown, Foias: *An elementary proof of a Theorem of Tacon.* Int. Eq. and Oper. Theory 4 (1981), 596-599.

[BK1] Buoni, Klein, Scott, Wadhwa: *On Power Compactness in a Banach Space.* Indiana Univ. Math. J. 32 (1983), 177-185.

[BK2] Buoni, Klein, Scott, Wadhwa: *Compact-like operators and the Baire Category Theorem.* Int. Eq. and Oper. Theory 7 (1984), 10-26.

[GM1] M. Gonzalez, A. Martinon: *On the generalized Sadovskii functor.* Rev. Acad. Canaria de Ciencias 1 (1990), 109-117.

[GM2] M. Gonzalez, A. Martinon: *Some extension of the Sadovskii functor.* Extracta Math. 5 (1990) 15-17.

[PI] A. Pietsch: *Operator ideals.* North-Holland; Amsterdam, New York, Oxford; 1980.

[TA] D. G. Tacon: *Two characterizations of power compact operators.* Proc. Amer. Math. Soc. 73 (1979), 356-360.