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A GENERALIZATION OF SEMI-FREDHOLM OPERATORS

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Several authors have introduced generalizations of the class SF_+ of upper semi-Fredholm operators [2, 5, 6, 7, 8, 12, 14]. SF_+ has good algebraic and topological properties: it is a semigroup (i.e., $ST \in SF_+$ when S and T belong to SF_+) which is stable under small norm perturbations (i.e., it is open). Also SF_+ is stable under perturbation by the class SS of the strictly singular operators. On the other hand, if $SF_+(X,Y)$ is nonempty, then the space X is essentially (up to finite dimensional subspace) isomorphic to a subspace of the space Y.

It is important to note that the definition and the properties of SF_+ are closely related with the class $\mathbb{F}$ of finite dimensional Banach spaces. In all the generalizations the underlying idea is to select another class of Banach spaces $\mathbb{A}$ (usually a space ideal in the sense of [9]), instead of $\mathbb{F}$, and then to define a class of operators $S\mathbb{A}_+$ which includes SF_+, and whose operators have kernel belonging to $\mathbb{A}$. And all that trying to preserve the nice properties of semi-Fredholm operators.

One of the applications of the generalized semi-Fredholm operators could be the classification of isomorphic properties of Banach spaces, since from the existence of an operator in $S\mathbb{A}_+(X,Y)$ it follows that X is (in some sense) isomorphic to a subspace of Y up to a subspace in $\mathbb{A}$.

With respect to the possible definitions of the generalized semi-Fredholm operators, if one only asks the kernel of the operator to be in $\mathbb{A}$, then the algebraic and topological properties of $S\mathbb{A}_+$ are not good [2], and it is not stable under perturbation.

On the other hand, it is too much to ask the ranges of the operators to be closed, since for non-semi-Fredholm operators this property is not stable under products or under compact perturbations of small norm [3; V.2.6].

The approach in [5, 6], using an operator ideal U, provides with classes of operators with good algebraic properties and stable under perturbations by operators in U, but in general these classes are not open [1,4].

In this paper, for every space ideal $\mathbb{A}$, we define generalized semi-Fredholm operators in a similar way to the approach to semi-Fredholm operators due to Schechter [11], Weis [13], Zemanek [15] and other authors, and show that its properties are better than that of the previous generalizations [2, 5, 6, 7, 8, 12, 14].

Recall that two Banach spaces X and Y are totally incomparable [10] if Banach spaces isomorphic to a subspace of X and to a subspace of Y have finite dimension. Given a space ideal $\mathbb{A}$ we shall denote $\mathbb{A}^{i}$ the class of Banach spaces which are totally incomparable with every space in $\mathbb{A}$.

$\mathbb{F}$ is the class of all finite dimensional Banach spaces; $\mathbb{A}$ is a space ideal in the sense of [9].

MAIN RESULTS. For $T \in L(X,Y)$, a continuous and linear operator from X into Y, we consider [9] $j(T) := inf\ \{\|Tx\| : \|x\|=1\}$, the injection modulus of T. If $X \notin \mathbb{A}$, then we define

$$sj_{\mathbb{A}}(T) := sup\ \{j(TJ_M) : M \subset X,\ M \notin \mathbb{A}\}$$
$$isj_{\mathbb{A}}(T) := inf\ \{sj_{\mathbb{A}}(TJ_M) : M \subset X,\ M \notin \mathbb{A}\}$$

Analogously, using the norm $n(T) := \|T\|$, we can define $in_{\mathbb{A}}$, $sin_{\mathbb{A}}$. We use the following characterizations: $T \in SF_+ \Leftrightarrow isj_{\mathbb{F}}(T)>0$, $T \in SS \Leftrightarrow sin_{\mathbb{F}}(T)=0$.

Definition. (1) $S\mathbb{A}_+(X,Y) := \{T \in L(X,Y) : isj_{\mathbb{A}}(T)>0\}$, if $X \notin \mathbb{A}$ and $Y \notin \mathbb{A}$,
$:= \varnothing$, the empty set, if $X \notin \mathbb{A}$ and $Y \in \mathbb{A}$,
$:= L(X,Y)$, if $X \in \mathbb{A}$.

(2) $\mathbb{A}SS(X,Y) := \{T \in L(X,Y) : isn_{\mathbb{A}}(T)=0\}$, if $X \notin \mathbb{A}$ and $Y \notin \mathbb{A}$,
$:= L(X,Y)$, if $X \in \mathbb{A}$ or $Y \in \mathbb{A}$.

Theorem. (a) $S\mathbb{A}_+$ is an open semigroup and $SF_+ \subset S\mathbb{A}_+$.

(b) $T \in S\mathbb{A}_+ \Rightarrow N(T) \in \mathbb{A}$

(c) $\mathbb{A}SS$ is closed and $SS \subset \mathbb{A}SS$.

(d) $\mathbb{A}=\mathbb{A}^{ii} \Rightarrow \mathbb{A}SS$ is a closed operator ideal.

(e) $Sp(\mathbb{A}SS) := \{X \in \mathbb{B} : I_X \in \mathbb{A}SS\} = \mathbb{A}$.

(e) $S\mathbb{A}_+$ is invariant by $\mathbb{A}SS$: $T \in S\mathbb{A}_+$, $S \in \mathbb{A}SS \Rightarrow T+S \in S\mathbb{A}_+$

EXAMPLES. In the following examples, we show that the semigroup $S\mathbb{A}_+$ can contain or not contain the operators with kernel in $\mathbb{A}$ and closed range. This give us an idea of the relative size of the semigroups.

1 Let $\mathbb{A}=\mathbb{N}l_1$: Banach spaces containing no copies of l_1. We show that in this case the semigroup $S\mathbb{A}_+$ is relatively broad. Let $T\in L(X,Y)$. If $R(T)$ is closed and $N(T)\in\mathbb{A}$, then $T\in S\mathbb{A}_+$.

If X be a Banach space such that the dual $X^*\in\mathbb{N}l_1$, we can construct an operator $T\in L(l_\infty,l_\infty)$ with kernel isomorphic to X^* and closed range. That show in particular that, for $\mathbb{A}=\mathbb{N}l_1$, the open semigroup $S\mathbb{A}_+(l_\infty,l_\infty)$ is strictly bigger than $SF_+(l_\infty,l_\infty)$.

2 Let $\mathbb{A}=l_2^{ii}$: Banach spaces such that every infinite dimensional subspace contains a copy of l_2. Contrarily to the case $\mathbb{A}=\mathbb{N}l_1$, in this case the semigroup $S\mathbb{A}_+$ does not contain all the operators with closed range and kernel in $\mathbb{A}$.

3 Let $\mathbb{A}=\mathbb{R}$: the class of all reflexive spaces. Recall that $T\in L(X,Y)$ is tauberian [7] when $T^{**-1}J_Y(Y)=J_X(X)$, where J_X is the canonical inclusion of X in X^{**}. The semigroup $S\mathbb{R}_+$ is properly contained in the class of tauberian operators. We do not know if operators with closed range and reflexive kernel belong to $S\mathbb{R}_+$. Note that these operators belong to the interior of the class of tauberian operators [1].

OBSERVATION. We have obtained an analogous generalization of the classes SF_- and SC of lower semi-Fredholm operators and strictly cosingular operators, respectively.

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