EXTRACTA MATHEMATICAE 5, n.2, 56-58 (1990)

OPERATIONAL QUANTITIES DERIVED FROM THE NORM AND MEASURES OF NONCOMPACTNESS

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Let X, Y be Banach spaces and $T: X \longrightarrow Y$ a (linear and continuous) operator. A.A. Sedaev [SE], and A. Lebow and M. Schechter [LS] introduced independently the quantity

$$c(T) := inf \left\{ \|TJ_M\| : M \in S(X),\ cod(M) < \infty \right\},$$

where $S(X)$ is the class of all (closed) subspaces of X, $J_M: M \longrightarrow X$ is the canonical inclusion and $cod(M)$ is the dimension of X/M. J. Zemanek [ZE] and A.S. Fajnshtejn and V.S. Shulman (see [FA]), introduced the quantity

$$d(T) := inf \left\{ \|Q_U T\| : U \in S(Y),\ dim(U) < \infty \right\},$$

where $Q_U: Y \longrightarrow Y/U$ is the quotient map and $dim(U)$ is the dimension of U. Moreover L.S. Goldstein, I.T. Gohberg and A.S. Markus (see [BG]) defined the measure of noncompactness $h(T)$ of an operator T, as the Hausdorff measure of noncompactness of the set TB_X, where B_X is the closed unit ball of X:

$$h(T) := inf \left\{ \varepsilon > 0 : \exists \{y_1, \ldots, y_n\} \subset Y,\ \forall x \in B_X,\ \exists i,\ \|Tx - y_i\| < \varepsilon \right\}.$$

The quantity h is surjective in the sense that $h(QT)=h(T)$, for every metric (linear) surjection $Q: Z \longrightarrow X$. H.-O. Tylli [TY] introduced another measure of noncompactness for operators, $t(T)$, given by

$$t(T) := inf \left\{ \varepsilon > 0 : \exists Z,\ \exists K: Z \longrightarrow X \text{ compact},\ \forall x \in X,\ \|Tx\| \le \|Kx\| + \varepsilon\|x\| \right\},$$

where Z is a Banach space. The quantity t is injective in the sense that $t(JT)=t(T)$, for every (linear) isometry $J: Y \longrightarrow Z$. We note that

$$T \text{ compact} \Leftrightarrow c(T)=0 \Leftrightarrow d(T)=0 \Leftrightarrow h(T)=0 \Leftrightarrow t(T)=0.$$

Suported in part by DGICYT Grant PB88-0417.

In this paper we show that $c=t$. A result of A.S. Fajnshtejn [FA] (also, independently, in [ET]) affirms that $d=h$. Hence both the injective and the surjective measure of noncompactness for operators can be derived from the norm. Moreover we consider two new measures of noncompactness for operators, ν and b. Both quantities are smaller than c and d, and it is not known if $\nu=b$. The quantity b is bijective (injective and surjective); hence its value does not depend on the size of the target space or of the source space. We use the quantity ν in order to prove that t, h, and ν are equivalent:

$$\nu \le \min\{c,d\} \le \max\{c,d\} \le 2\nu ,$$

and we also study the relation between the values of the quantities for an operator and the values for its conjugate operator.

Related results and a systematic study of operational quantities can be found in [MA].

MAIN RESULTS

Theorem 1. *$t=c$ and $h=d$.*

In addition to the quantities c and d, another related quantity can be derived from the norm:

$$\nu(T) := \inf \left\{ \|Q_U T J_M\| : U\in S(Y),\ \dim(U)<\infty\},\ M\in S(X),\ \mathrm{cod}(M)<\infty \right\}$$

Theorem 2. *$\nu\le c=t\le 2\nu$ and $\nu\le d=h\le 2\nu$. Consequently, $t\le 2h$* and *$h\le 2t$.*

We obtain now several duality results. If $T\colon X \longrightarrow Y$ is an operator, then $T^*\colon Y^* \longrightarrow X^*$ is the conjugate operator.

Proposition 3. *(1) $c(T) = d(T^*)$ (2) $c(T^*) \le d(T)$*
(3) $t(T) = h(T^*)$ (4) $t(T^*) \le h(T)$
(5) $h(T^{**}) \le h(T)$ (6) $t(T^{**}) \le t(T)$
(7) $h(T) \le 2h(T^*) \le 4h(T)$ (8) $t(T) \le 2t(T^*) \le 4t(T)$

The results (3), (4) and (5) are in [SE]; (4) in [GM]. It is not known if ν is a surjective or an injective quantity. However it is possible to prove these properties for a related quantity b that we shall introduce below.

We give expresions for the quantities c and d, which will suggest the definition of b. In order to do that, we recall several concepts. For every Banach space X we consider $X^{inj}:=l_1(B_X)$. The natural map $Q_X\colon X^{inj} \longrightarrow X$ is a metric surjection [PI; C.3.3]. On the other hand, if $Y^{sur}:=l_\infty(B_{Y^*})$, then the natural map $J_Y: Y \longrightarrow Y^{sur}$ is a metric injection [PI; C.3.7].

Moreover, for every operator $T: X \longrightarrow Y$, we define the quantity

$$a(T) := \inf \{\|T-F\| : F: X \longrightarrow Y \text{ operator}, \dim R(F) < \infty\} .$$

a is the distance from T to the class of finite dimensional operators.

The following proposition relates the quantities c, d and a.

Proposition 4. $c(T) = a(J_Y T)$ *and* $d(T) = a(TQ_X)$

The expressions in the proposition 4 suggest introducing for every operator $T: X \longrightarrow Y$ the following quantity

$$b(T) := a(J_Y T Q_X) .$$

Proposition 5. *b is bijective and* $b(T) \leq \min \{c(T),d(T)\}$.

It is not known if $b(T)=\nu(T)$ for every operator T. Moreover, we can derive from the quantities $c=t$, $d=h$ and ν other quantities which can be used to characterize the classes of semi-Fredholm, strictly singular, strictly cosingular and compact operators (see [MA]).

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