SOME NEW CLASSES OF OPERATORS BETWEEN BANACH SPACES
SEMITAUBERIAN OPERATORS

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We are going to define some classes of operators in the same way as semi-Fredholm and tauberian operators are defined. An operator $T \in \mathcal{L}(X,Y)$ is semi-Fredholm if for every bounded set $A$ of $X$ such that $T(A)$ is a relatively compact set of $Y$, $A$ is relatively compact. If we change in this definition relatively compact by weakly relatively compact we have got the definition of tauberian operator, and if we change it for weakly conditionally compact we have got the definition of semi-tauberian operator (see [3]). In the following we are going to study some properties of these operators when we use at the definition a family of sets with the following properties:

Definition: We will call $P_0$ to a family of sets with the following properties:

- For every Banach space $X$ it verifies that $P_0(X)cB(X)=$ (bounded sets), and there is a Banach space $Y$ such that $(\emptyset)\in P_0(Y)cB(Y)$.

- For every $X$ and $Y$ Banach spaces, $T \in \mathcal{L}(X,Y)$ and $A \in P_0(X)$ we have that $T(A) \in P_0(Y)$.

- For every Banach space $X$ and $A, B \in P_0(X)$ we have that $A+B \in P_0(X)$.

- For every Banach space $X$, $A \in P_0(X)$ and $A \subseteq A$ it follows that $A \in P_0(X)$.

Remark: From the definition we see that for every Banach space $X$ $B(X)=P_0(X)$ if and only if $B_X=$(the unit ball of $X) \in P_0(X)$.

Definition: Given a family of sets $P_0$, an operator $T \in \mathcal{L}(X,Y)$ has the $P_0$-property if for every bounded set $A$ of $X$ such that $T(A) \in P_0(Y)$ it follows that $A \in P_0(X)$. We will denote the family of operators between $X$ and $Y$ with the $P_0$-property by $P_0(X,Y)$.

One of the reason of defining these operators is the following property. If $T \in P_0(X,Y)$ and there is other family $P_0'$ such that $P_0'(Y)cP_0(Y)$ then $P_0'(X)cP_0(X)$. For example if $T \in \mathcal{L}(X,Y)$ is

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semi-Fredholm and \( Y \) has Gelfand-Phillips property (resp [1]) (resp Schur property) then \( X \) has Gelfand-Phillips property (resp. Schur property). If \( T \in \mathcal{E}(X,Y) \) is tauberian and \( Y \) is reflexive (resp \( Y \) has the reciprocal Dunford-Pettis property (see [1])) then \( X \) is reflexive (resp. \( X \) has the reciprocal Dunford-Pettis property).

Some cases where we can simplify the definition of the operators with the \( P_0 \)-property

It is obvious that if \( P_0 \) and \( P_0' \) are two families of sets such that \( P_0'(Y) \subseteq P_0(Y) \) (for example the relatively compact sets are contained in a lot of families of sets) then every operator \( T \in \mathcal{E}(X,Y) \) verifies that if \( A \) is a bounded set with \( T(A) \subseteq P_0'(Y) \) then \( A \subseteq P_0(X) \). This property and other similar properties (inspired by the paper [2]) that I will give in the following lemma, are sufficient for an operator to have the \( P_0 \)-property when \( P_0 = \mathcal{W}(\text{weakly relatively compact sets}) \) or \( P_0 = \mathcal{W}(\text{weakly conditionally compact sets}) \) or \( P_0 = \mathcal{D} \) (Dunford-Pettis sets) or \( P_0 = \mathcal{L}(\text{limited sets}) \) (see [1]).

**Lemma 1:** If \( T \in \mathcal{E}(X,Y) \) and \( P_0' \) is another family of sets such that \( P_0'(Y) \subseteq P_0(Y) \) then \( T \) has the following properties:

1. For every \( A \in B(X) \) such that \( T(A) \subseteq P_0'(Y) \) it verifies that \( A \subseteq P_0(X) \).

2. For every \( Z \) Banach space, \( S \in \mathcal{E}(Z,X) \) such that \( T(S(B_Z)) \subseteq P_0'(Y) \), it verifies that \( S(B_Z) \subseteq P_0(X) \).

3. For every closed subspace \( H \) of \( X \) if \( T(B_h) \subseteq P_0'(Y) \) then \( B_h \subseteq P_0(X) \).

4. For every \( S \in \mathcal{E}(X,Y) \) such that \( S(B_k) \subseteq P_0'(Y) \) it verifies that \( T \subseteq \mathcal{E}(X,Y) \).

5. For every \( S \in \mathcal{E}(X,Y) \) such that \( S(B_k) \subseteq P_0'(Y) \) it verifies that \( B_{k \in \mathcal{T}} \subseteq P_0(X) \).

**Proof:** Every property is immediately; besides, it is easy to see that 1)\( \Rightarrow \) 2)\( \Rightarrow \) 3)\( \Rightarrow \) 4)\( \Rightarrow \) 5).

**Theorem 1:** If \( P_0 = \mathcal{W}, \mathcal{D}, \mathcal{L} \) or \( \mathcal{E}(X,Y) \) verifies that for every \( S \in \mathcal{E}(X,Y) \) compact operator (i.e. \( S(B_k) \subseteq \mathcal{R}(Y) \)) \( B_{k \in \mathcal{T}} \subseteq \mathcal{P}_0(X) \) then \( T \in \mathcal{E}(X,Y) \).

After the last theorem it is easy to see that if \( P_0 = \mathcal{W}, \mathcal{D}, \mathcal{L} \) or \( \mathcal{E}(X,Y) \) and \( P_0'' \) is a family of sets such that for every Banach space \( X \) \( \mathcal{R}(X) \subseteq P_0''(X) \subseteq P_0(X) \) then every property from 1) to 5) of the lemma 1 is
equivalent to that $TeP_0(X,Y)$ when we take $P'_0 = P_0''$. For example if
$P_0 = WC$ (i.e. when $T$ is a semi-tauberian operator) we can use in the
lemma 1 $P'_0 = WC$, $W$ or $RC$ and the five properties are necessary and
sufficient so that $T$ is semi-tauberian. So, for example, $Te\mathbb{E}(X,Y)$ is
semi-tauberian if and only if for every operator $Se\mathbb{E}(X,Y)$ compact,
weakly compact or Rosenthal, $T+S$ is semi-tauberian.

In the following result we are going to see that the property 1)
of the lemma 1 is very interesting when we take $P'_0 = RC$.

Definition: A family of sets $P_0$ is called $P_{00}$ when it has the
following properties:

1) For every Banach spaces $X$, $A \in P_0(X)$ if and only if for every $c > 0$
there is a set $B \in P_0(X)$ such that $A \in B + cB_x$.

2) For every Banach space $X$ $P_0(X)$ is closed by finite unions.

Theorem 2: Given a family of sets $P_{00}$ and an operator $Te\mathbb{E}(X,Y)$
with closed range the following two assertions are equivalents:

1) $B_{x,y} \in P_{00}(X)$

ii) For every set $A \in B(X)$ such that $T(A) \in RC(Y)$ it follows that
$A \in P_{00}(X)$.

So when $P_{00} = WC, W, DP$ or $L$ the last lemma proves that if $Te\mathbb{E}(X,Y)$
has a closed range and $B_{x,y} \in P_{00}(X,Y)$ then $TeP_{00}(X,Y)$. This property is
specially interesting when $T$ is a projection over a quotient space
because it lets us to know the limited sets of $X$, for example, knowing
their images in the quotient space (if the unit ball of the subspace is
a limited set).

REFERENCES

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