EXTRACTA MATHEMATICAE 5, n.2, 53-55 (1990)

SOME NEW CLASSES OF OPERATORS BETWEEN BANACH SPACES
SEMITAUBERIAN OPERATORS

Beatriz Hernando *
Dpto. Anàlisis Matemàtico. Facultad de Matemàticas.
Universidad Complutense. 28040 Madrid.
AMS classification (1985) 47A05, 47A65, 47D05.

We are going to define some classes of operators in the same way as semi-Fredholm and tauberian operators are defined. An operator $T\in\mathcal{L}(X,Y)$ is semi-Fredholm if for every bounded set A of X such that T(A) is a relatively compact set of Y, A is relatively compact. If we change in this definition relatively compact by weakly relatively compact we have got the definition of tauberian operator, and if we change it for weakly conditionally compact we have got the definition of semi-tauberian operator (see [3]). In the following we are going to study some properties of these operators when we use at the definition a family of sets with the following properties:

Definition: We will call P_0 to a family of sets with the following properties:

- For every Banach space X it verifies that $P_0(X)\subset B(X)=\{$bounded sets$\}$, and there is a Banach space Y such that $\{\emptyset\}\neq P_0(Y)\subseteq B(Y)$.

- For every X and Y Banach spaces, $T\in\mathcal{L}(X,Y)$ and $A\in P_0(X)$ we have that $T(A)\in P_0(Y)$.

- For every Banach space X and $A,B\in P_0(X)$ we have that $A+B\in P_0(X)$.

- For every Banach space X, $A\in P_0(X)$ and $A_0\subset A$ it follows that $A_0\in P_0(X)$.

Remark: From the definition we see that for every Banach space X $B(X)=P_0(X)$ if and only if $B_X=\{$the unit ball of X$\}\in P_0(X)$.

Definition: Given a family of sets P_0, an operator $T\in\mathcal{L}(X,Y)$ has the P_0-property if for every bounded set A of X such that $T(A)\in P_0(Y)$ it follows that $A\in P_0(X)$. We will denote the family of operators between X and Y with the P_0-property by $P_0(X,Y)$.

One of the reason of defining these operators is the following property. If $T\in P_0(X,Y)$ and there is other family P_0' such that $P_0'(Y)\subset P_0(Y)$ then $P_0'(X)\subset P_0(X)$. For example if $T\in\mathcal{L}(X,Y)$ is

* Supported in part by DGICYT-PB88-0141

semi-Fredholm and Y has Gelfand-Phillips property (see [1]) (resp Schur property) then X has Gelfand-Phillips property (resp. Schur property). If $T\in\mathcal{L}(X,Y)$ is tauberian and Y is reflexive (resp Y has the reciprocal Dunford-Pettis property (see [1])) then X is reflexive (resp. X has the reciprocal Dunford-Pettis property).

Some cases where we can simplify the definition of the operators with the P_0-property

It is obvious that if P_0 and P_0' are two families of sets such that $P_0'(Y)\subset P_0(Y)$ (for example the relatively compact sets are contained in a lot of families of sets) then every operator $T\in P_0(X,Y)$ verifies that if A is a bounded set with $T(A)\in P_0'(Y)$ then $A\in P_0(X)$. This property and other similar properties (inspired by the paper [2]) that I will give in the following lemma, are sufficients for an operator to have the P_0-property when P_0=W={weakly relatively compact sets} or P_0=WC={weakly conditionally compact sets} or P_0=DP={Dunford-Pettis sets} or P_0=L={limited sets} (see [1]).

Lemma 1: **If $T\in P_0(X,Y)$ and P_0' is other family of sets such that $P_0'(Y)\subset P_0(Y)$ then T has the following properties:**

i) For every $A\in B(X)$ such that $T(A)\in P_0'(Y)$ it verifies that $A\in P_0(X)$.

ii) For every Z Banach space, $S\in\mathcal{L}(Z,X)$ such that $TS(B_Z)\in P_0'(Y)$, it verifies that $S(B_Z)\in P_0(X)$.

iii) For every closed subspace H of X if $T(B_H)\in P_0'(Y)$ then $B_H\in P_0(X)$.

iv) For every $S\in\mathcal{L}(X,Y)$ such that $S(B_X)\in P_0'(Y)$ it verifies that $T+S\in P_0(X,Y)$.

v) For every $S\in\mathcal{L}(X,Y)$ such that $S(B_X)\in P_0'(Y)$ it verifies that $B_{KerT+S}\in P_0(X)$.

Prof: Every property is immediately; besides, it is easy to see that i)⇒ii)⇒iii)⇒v) and iv)⇒v). ∎

Theorem 1: **If P_0=W, WC, DP or L and $T\in\mathcal{L}(X,Y)$ verifies that for every $S\in\mathcal{L}(X,Y)$ compact operator (i.e. $S(B_X)\subset RC(Y)$) $B_{KerT+S}\in P_0(X)$ then $T\in P_0(X,Y)$.**

After the last theorem it is easy to see that if P_0=W, WC, DP or L and P_0'' is a family of sets such that for every Banach space X $RC(X)\subset P_0''(X)\subset P_0(X)$ then every property from i) to v) of the lemma 1 is equivalent to that $T\in P_0(X,Y)$ when we take $P_0'=P_0''$. For example if $P_0=WC$ (i.e. when T is a semi-tauberian operator) we can use in the lemma 1 $P_0'=WC$, W or RC and the five properties are necessary and sufficient so that T is semi-tauberian. So, for example, $T\in\mathcal{L}(X,Y)$ is semi-tauberian if and only if for every operator $S\in\mathcal{L}(X,Y)$ compact, weakly compact or Rosenthal, T+S is semi-tauberian.

In the following result we are going to see that the property i) of the lemma 1 is very interesting when we take $P_0'=RC$.

Definition: A family of sets P_0 is called P_{00} when it has the following properties:

1-For every Banach spaces X, $A\in P_0(X)$ if and only if for every $\varepsilon>0$ there is a set $B\in P_0(X)$ such that $A\subset B+\varepsilon B_X$.

2-For every Banach space X $P_0(X)$ is closed by finite unions.

Theorem 2: **Given a family of sets P_{00} and an operator $T\in\mathcal{L}(X,Y)$ with closed range the following two assertions are equivalents:**

i)$B_{KerT}\in P_{00}(X)$

ii)For every set $A\in B(X)$ such that $T(A)\in RC(Y)$ it follows that $A\in P_{00}(X)$.

So when $P_{00}=WC$, W, DP or L the last lemma proves that if $T\in\mathcal{L}(X,Y)$ has a closed range and $B_{KerT}\in P_{00}(X,Y)$ then $T\in P_{00}(X,Y)$. This property is specially interesting when T is a projection over a quotient space because it lets us to know the limited sets of X, for example, knowing their images in the quotient space (if the unit ball of the subspace is a limited set).

REFERENCES

[1] Bombal F. "Sobre algunas propiedades de espacios de Banach" Rev. Acad. de Ciencias de Madrid (to appear)

[2] Gonzalez M. and Onieva V.M. "Characterizations of tauberian operators and other semigroups of operators".Proc. American Math. Soc. Vol. 108. Number 2 (1990)

[3] Hernando B. "A new class of operators between Banach spaces" Extracta Math. (to appear).