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# THEOREMS OF ε-PSEUDOORTHOGONAL DECOMPOSITION IN NORMED LINEAR SPACES.

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Let X be a normed linear space over real or complex number field K. The following definition is a natural generalization of Birkhoff-James' orthogonality in normed linear spaces [1], [3].

**DEFINITION 1.** Let $\varepsilon\in[0,1)$. The element $x\in X$ will be called ε-Birkhoff-James pseudoorthogonal on the element $y\in X$ or ε-B-J-pseudoorthogonal, for short, if we have $\|x+\lambda y\|\geq(1-\varepsilon)\|x\|$ for all $\lambda\in K$. We denote this $x\perp_{\varepsilon}y$ (B-J).

If A is a nonempty subset in X, then by $A^{\perp_{\varepsilon}}$(B-J) we denote the set of all elements which are ε-B-J-pseudoorthogonal over A. We remark that $0\in A^{\perp_{\varepsilon}}$(B-J) and $A\cap A^{\perp_{\varepsilon}}$(B-J) $\subseteq\{0\}$ for every $\varepsilon\in[0,1)$.

The following statement is a variant of F. Riesz' result (see e.g. [4], p. 84):

**PROPOSITION 1.** Let X be as above and E be its closed linear subspace. Suppose $E\neq X$. Then for every $\varepsilon\in(0,1)$ the set $E^{\perp_{\varepsilon}}$(B-J) is nonzero.

Now, we can give the first ε-pseudoorthogonal decomposition theorem which works in normed linear spaces.

**THEOREMS 1.** Let X be a normed space and E be its closed linear subspace. Then for every $\varepsilon\in(0,1)$ the following decomposition holds:

$$X = E + E^{\perp_{\varepsilon}}(B\text{-}J).$$

Indeed, it is clear that for $E\neq X$ and $x\notin E$, there exists an element $y_{\varepsilon}\in E$ such that $0<d=d(x,E)\leq\|x-y_{\varepsilon}\|\leq d/(1-\varepsilon)$ and since $x_{\varepsilon}:=x-y_{\varepsilon}\in E^{\perp_{\varepsilon}}$(B-J) we obtain the desired representation.

A mapping $[\ ,\ ]:X\times X\longrightarrow K$ is called semi-inner product (s.i.p.) on linear space X if the following conditions are satisfied:

(i) $[x,x]\geq 0$ for all $x\in X$ and $[x,x]=0$ implies $x=0$;

(ii) $[\lambda x,y]=\lambda[x,y]$ and $[x,\lambda y]=\bar{\lambda}[x,y]$ for all $x,y\in X$;

(iii) $|[x,y]|^2\leq [x,x][y,y]$ for all x,y in X;

(iv) $[x+y,z]=[x,z]+[y,z]$ for all x,y,z in X.

It is clear that the mapping $x\in X \longrightarrow [x,x]^{1/2}\in \mathbb{R}_+$ is a norm on X and every s.i.p. on normed space X which generates its norm is of the form: $[x,y]=\langle\tilde{J}(y),x\rangle$ for all x,y in X where $\tilde{J}$ is a section of normalized dual mapping.

The following concept is a generalization in one sense of Giles' orthogonality [2]:

**DEFINITION 2.** Let $\varepsilon\in[0,1)$. The element $x\in X$ is called $\varepsilon$-Giles pseudo-orthogonal on the element $y\in X$ (relative to s.i.p. [ , ]) or $\varepsilon$-G-pseudo-orthogonal, for short, if $|[y,x]|\leq\varepsilon\|x\|\ \|y\|$ and we denote $x\perp_{\varepsilon}y(G)$.

If A is a nonempty subset of X, then by $A^{\perp_{\varepsilon}}(G)$ we shall denote the set of all elements which are $\varepsilon$-G-pseudoorthogonal over A. It is easy to see that $0\in A^{\perp_{\varepsilon}}(G)$ and $A\cap A^{\perp_{\varepsilon}}(G)\subseteq\{0\}$ for all $\varepsilon\in[0,1)$.

**PROPOSITION 2.** Let $(X;(\ ,\ ))$ be an inner-product space and $\varepsilon\in[0,1)$. Then $x\perp_{\varepsilon}y(G)$ iff $x \perp_{\eta(\varepsilon)} y(B\text{-}J)$ where $\eta(\varepsilon):=1-(1-\varepsilon^2)^{1/2}$.

The proof follows by the properties of i.p. and we omit it.

In virtue of this fact, we can introduce the following concept.

**DEFINITION 3.** Let X be a normed space and [ , ] be a s.i.p. on it which generates its norm. The s.i.p. [ , ] will be called of (APP)-type if there exists a mapping $\eta:[0,1) \longrightarrow [0,1)$ (called the transition mapping) such that $\eta(\varepsilon)=0$ iff $\varepsilon=0$ and with the property that $x \perp_{\eta(\varepsilon)} y(B\text{-}J)$ implies $x\perp_{\varepsilon}y(G)$, for all $\varepsilon\in(0,1)$.

It is clear that every i.p. on a linear space X is a s.i.p. of (APP)-type. Now, let $(\Omega,\mathcal{A},\mu)$ be a measure space and $L^p(\Omega)$ $(p>1)$ be the Banach space of p-integrables real functions on $\Omega$. If we put $(y,x)_p:=\lim_{t\to 0}(\|x+ty\|_p^2-\|x\|_p^2)/2t$ for x, y in $L^p(\Omega)$, $p\geq 2$, then $(\ ,\ )_p$ is a s.i.p. of (APP)-type with the transition mapping $\eta(\varepsilon):=1-[1-\varepsilon^2/(2p-3)]^{1/2}$, $\varepsilon\in[0,1)$.

**PROPOSITION 3.** Let X be a normed space and [ , ] be a s.i.p. of (APP)-type which generates its norm. If E is a proper closed linear subspace in X then $E^{\overset{\perp}{\varepsilon}}(G)$ is nonzero for all $\varepsilon \in (0,1)$.

**COROLLARY.** If $X_p$ is a nonzero linear subspace in $L^p(\Omega)$ $(p \geq 2)$ and $E_p$ is a proper closed linear subspace in $X_p$, then $E_p^{\overset{\perp}{\varepsilon}}(G)$ is nonzero for all $\varepsilon \in (0,1)$ where $E_p^{\overset{\perp}{\varepsilon}}(G)$ is taken in $X_p$.

The following $\varepsilon$-pseudoorthogonal decomposition theorem also holds.

**THEOREM 2.** Let X and E be as above. Then for all $\varepsilon \in (0,1)$ we have the decomposition:

$$X = E + E^{\overset{\perp}{\varepsilon}}(G) .$$

**COROLLARY.** If $X_p$ and $E_p$ are as above, then for any $\varepsilon \in (0,1)$ we have the decomposition:

$$X_p = E_p + E_p^{\overset{\perp}{\varepsilon}}(G) .$$

## REFERENCES

[1]. G. BIRKHOFF, Orthogonality in linear metric spaces. Duke Math. J., 1 (1935), 169-172.

[2]. J.R. GILES, Classes of semi-inner-product spaces. Trans. Amer. Mat. Soc., 129 (1967), 436-446.

[3]. R.C. JAMES, Orthogonality and linear functionals in normed linear spaces. Trans. Amer. Math. Soc., 61 (1947), 265-292.

[4]. K. YOSIDA, Functional Analysis. Springer-Verlag, Berlin 2-nd Ed., 1968.