ABOUT CERTAIN ISOMORPHIC PROPERTIES OF BANACH SPACES IN PROJECTIVE TENSOR PRODUCTS

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This note is an announcement of results contained in the papers [4],[5],[6] concerning isomorphic properties of Banach spaces in projective tensor products (for this definition and some property we refer to [1]). At the end, some new result is obtained, too.

In the sequel $L(E,F^*)$ (resp. $K(E,F^*)$) denotes the space of all operators (resp. compact operators) from a Banach space $E$ into the dual Banach space $F^*$. The first result needs the definition of a new isomorphic property introduced in [4]: we say that a Banach space $X$ has the (DPrcP) if any Dunford-Pettis set, i.e., a bounded set $M$ such that $\lim_{n} \sup_{x \in M} |x^n(x)| = 0$ for any $\omega$-null sequence $(x^n) \subset X^*$, is relatively compact. In [4] we obtained the following results

**THEOREM 1** [4]. A dual Banach space $X^*$ has the (DPrcP) iff $X$ does not contain a copy of $l^1$.

**THEOREM 2** [4]. Let $E,F$ be two Banach spaces not containing $l^1$. If $L(E,F^*) = K(E,F^*)$, then $E \subset F$ does not contain $l^1$.

The proof of Theorem 2 is based upon Theorem 1 and another characterization of Banach spaces not containing $l^1$ proved in [3].

We note that Theorem 2 answers a question put by Ruesch in [12].

The following two results we present are about Property (V) of Pelczynski ([10]) and the Reciprocal Dunford-Pettis Property (RDPP) ([8]). The first is from [5], the other from [6].

**THEOREM 3.** Let $E$ be a Banach space with Property (V) and $F$ be a reflexive space. If $L(E,F^*) = K(E,F^*)$, then $E \subset F$ has Property (V).

**THEOREM 4.** Let $E,F$ be two Banach spaces. Then the following are equivalent, provided $L(E,F^*) = K(E,F^*)$,

i) $E$ and $F$ possess the (RDPP) and $l^1$ doesn't embed into at least one of them

ii) $E \subset F$ has the (RDPP).

In the proof of Theorem 3 we used a characterization of Property (V) by Pelczynski in [10].

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property (V) contained in [10] and in the proof of Theorem 4 a characterization of the (RDPP) to be found in [9] and the already quoted result from [3]. We note that all of our papers contain remarks about the necessity of the assumption "L(E,F*) = K(E,F*)". In order to illustrate the techniques we used, based upon results about weak sequential compactness in K(E,F*) ([12]), we present a new result about the so-called Grothendieck Property (GrP): a B-space X has the (GrP) if w*-null sequences in X* are w-null ([11]).

THEOREM 5. Let E be a B-space with the (GrP) and F be a reflexive space. If L(E,F*) = K(E,F*), then $E_{\pi}F$ has the (GrP).

Proof. Let $(b_n)$ be a w*-null sequence in K(E,F*) = $(E_{\pi}F)^*$. Take $x^*yE^*$ and $yF^*$. The operator mapping $B \in K(E,F^*)$ into $B^*(y)E^*$ is w*-w* sequentially continuous; indeed, if $(T_n)$ is w*-null and $x \in E$, we have $T_n^*(y)(x) = T_n(x y) \to 0$ because $x y \in E_{\pi}F$ and $(T_n)$ is w*-null. Hence $B_n^*(y) \wto 0$ in E*. But E has the (GrP) and so $B_n^*(y) \wto 0$. Hence, $B_{**}(x^{**})(y) \to 0$ and this means that $B_n \wto 0$ ([12]). We are done.

THEOREM 6. Let E be a reflexive space and F be a B-space with the (GrP). If L(E,F*) = K(E,F*), then $E_{\pi}F$ has the (GrP).

Proof. Since $E_{\pi}F$ is isomorphic to $F_{\pi}E$, it is enough to apply Theorem 5 to $F_{\pi}E$; so we need to prove that L(F,E*) = K(F,E*).

Take $T \in L(F,E*)$; we have $T^*: E^* \to F^*$ and $T_{E^*} \in K(E,F^*)$. Since $T^*$ is w*-w* continuous and $B_{E^*}$ is w*-dense in $B_{E^{**}}$, it is quite easy to prove that $T_{E^*}(B_{E^*}) \to T^*(B_{E^{**}})$. We are done.

The hypothesis of reflexivity of E(or F) in the above results is not restrictive thanks to the following remark.

REMARK. If $E_{\pi}F$ has the (GrP), then either E or F is reflexive.

Assume $l^1$ embeds into both E and F. A result in [11] gives that ($l^1$ and so $l^2$ embeds into both $E^*$ and $F^*$. Hence $l^2_{\pi}l^1$ is a subspace of L(E,F*), that is weakly sequentially complete as a dual of a space with the (GrP) must be. Now, recall that $c_0$ lives inside of $l^2_{\pi}l^1$; a contradiction. And so either E or F contains $l^1$; this means that either E or F is reflexive ([12]).
As a consequence, we note that the space $l^\infty_{\mathbb{R}}$, $1^{\infty}$ cannot have the (GrP), whereas $l^\infty_{\mathbb{R}}$, $1^p$, $2 < p < \infty$, has that property. Instead, $l^\infty_{\mathbb{R}}$, $1^p$, $1 < p \leq 2$, doesn't possess the (GrP), since its dual space contains $c_0$, as proved in [7].

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