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DISTINGUISHED SUBSETS AND COMPLEMENTED COPIES OF c_0 IN VECTOR SEQUENCE SPACES

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Recall that a subset K of a Banach space E is said to be *limited* (resp., *Dunford-Pettis)* if for every weak* null (resp., weakly null) sequence (x_n^*) in the topological dual E^* of E, the following holds:

$$\lim_{n\to\infty} \sup\{|x_n^*(x)| \ : \ x\in K\} = 0.$$

(See [BD] and [A].)

The classes $\mathcal{L}(E)$ and $\mathcal{DP}(E)$ of limited and Dunford-Pettis subsets of E, respectively, are formed by bounded sets, and they are preserved by continuous linear images, linear combinations, closed absolutely convex hulls and passing to subsets. Also the following relationships hold

$$(\dagger)\qquad \begin{array}{l} \mathcal{K}(E) \subset \mathcal{L}(E) \subset \mathcal{DP}(E) \subset \mathcal{WC}(E) \\ \qquad\qquad\qquad \mathcal{K}(E) \subset \mathcal{W}(E) \subset \mathcal{WC}(E) \end{array}$$

where $\mathcal{K}(E)$, $\mathcal{W}(E)$ and $\mathcal{WC}(E)$ stand for the relatively norm compact, weakly relatively compact and weakly conditionally compact subsets of E, respectively (recall that $A \in \mathcal{WC}(E)$ if every sequence in A has a weakly Cauchy subsequence. Note that the proof of $\mathcal{L}(E) \subseteq \mathcal{WC}(E)$ given in [DR] also works to show that $\mathcal{DP}(E) \subseteq \mathcal{WC}(E)$.)

We are now interested in the study of the different classes of distinguished subsets ($\dagger$) in spaces of vector sequences.

Let (E_n) be a sequence of Banach spaces and $1\le p<\infty$. We shall denote, as usual, by $(\sum\oplus E_n)_p$ the space of all vector valued sequences $x = (x_n)$ such that $x_n\in E_n$ $(n\in\mathbb{N})$ and $\|x\|_p^p = \sum_{n=1}^{\infty} \|x_n\|^p$ is finite, endowed with the Banach norm $x \longmapsto \|x\|_p$.

Reasoning as in the scalar case, it is very easy to prove the following (well known) proposition (see [BrD], Th 2 or [B], Prop. 8.)

Proposition 1. *Let (E_n) be a sequence of Banach spaces and $1\le p<\infty$. For a* *bounded subset* $A \subseteq E = (\sum \oplus E_n)_p$, *the following propierties are equivalent:*

a) $A \in \mathcal{W}(E)$ (*resp.* $A \in \mathcal{WC}(E)$).

b) *We have*

i) *For every* $k \in \mathbb{N}$, $\pi_k(A) \in \mathcal{W}(E_k)$ (*resp.*, $\mathcal{WC}(E_k)$)

and

ii) *If* $p = 1$, *the following condition holds:*

$$\lim_{n\to\infty} \sup \left\{ \sum_{k=n}^{\infty} \|\pi_k(x)\| \ : \ x \in A \right\} = 0.$$

For the other classes in (†), we have

Proposition 2 *Let* (E_n) *be a sequence of Banach spaces,* $\mathcal{H}$ *any of the classes* $\mathcal{K}$, $\mathcal{L}$ *or* $\mathcal{DP}$, *and* $1 \le p < \infty$. *For a bounded subset* $A \subseteq E = (\sum \oplus E_n)_p$, *the following properties are equivalent:*

a) $A \in \mathcal{H}(E)$.

b) *We have*

i) *For each* $k \in \mathbb{N}$, $\pi_k(A) \in \mathcal{H}(E_k)$.

ii) $\lim_{n\to\infty} \sup \left\{ \sum_{k=n}^{\infty} \|\pi_k(x)\|^p \ : \ x \in A \right\} = 0.$

Many important properties of a Banach space E are (or can be) defined by the coincidence of two classes of distinguished subsets of E. For example:

-E has the *Dunford-Pettis Property* if $\mathcal{WC}(E) = \mathcal{DP}(E)$ ([A]).

-E has the *Gelfand-Phillips Property* if $\mathcal{L}(E) = \mathcal{K}(E)$ ([BD]).

-E has the *Schur Property* if $\mathcal{K}(E) = \mathcal{W}(E)$.

-E is *weakly sequentially complete* if $\mathcal{WC}(E) = \mathcal{W}(E)$.

-E has the *RDP** *Property* if $\mathcal{DP}(E) \subseteq \mathcal{W}(E)$ ([L]).

Hence propositions 1 and 2 can be interpreted as stability results for such properties when passing from the sequence (E_n) to the corresponding ℓ_p-sum:

Corollary 3. *Let* (E_n) *be a sequence of Banach spaces and* $F_p = (\sum \oplus E_n)_p$ ($1 \le p < \infty$).

a) *On* F_1 *two of the classes* $\mathcal{WC}$, $\mathcal{W}$, $\mathcal{DP}$, $\mathcal{L}$ *or* $\mathcal{K}$ *coincide if and only if they coincide on every* E_n. *In particular,* F_1 *is weakly sequentially complete (resp., has the Schur property, the Dunford-Pettis property, the Gelfand-Phillips property or the RDP***-property) if and only if so does every* E_n.

b) *On* F_p ($1 < p < \infty$) *two of the classes* $\mathcal{K}$, $\mathcal{L}$ *or* $\mathcal{DP}$ *coincide if and only if they coincide on every* E_n. *Also* $\mathcal{W}(F_p) = \mathcal{WC}(F_p)$ *if and only if* $\mathcal{W}(E_n) = \mathcal{WC}(E_n)$, *for every n. In particular,* F_p *is weakly sequentially complete (resp., has the Gelfand-Phillips property or the RDP** *property)* *if and only if so does every E_n.*

Limited sets are especially useful for detecting complemented copies of c_o, due to the following result:

Lemma 4. *([SL], [E1]) A Banach space contains a complemented copy of c_o if and only if it contains a non limited sequence (x_n), equivalent to the unit c_o-basis.*

By using the above result, Emmanuele proved in [E2] that if μ is a non purely atomic, finite measure and E contains a (non necessarily complemented) copy of c_o, the space $L_p(\mu,E)$ $(1\le p<\infty)$ of all E-valued Bochner μ-integrable functions, contains always a *complemented* copy of c_o. For a purely atomic measure, the situation is completely different, as we shall si as a consequence of the following result:

Theorem 5. *Let (E_n) be a sequence of Banach spaces, $1\le p<\infty$ and $F_p = (\sum\oplus E_n)_p$. Then F_p contains a complemented copy of c_o if and only if there exists some $n \in \mathbb{N}$ such that E_n contains a complemented copy of c_o.*

Corollary 6. *Let μ be a σ-finite, purely atomic measure, $1\le p<\infty$ and E a Banach space. Then $L_p(\mu,E)$ contains a complemented copy of c_o if and only if E contains a complemented dopy of c_o.*

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