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# PROPORTIONAL FACTORIZATION OF ENTROPY IDEALS

by

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A classical paper of A.Pietsch [P] establishes the identities $\mathfrak{A}_r=\mathfrak{A}_p\circ\mathfrak{A}_q$ and $\mathfrak{E}_r=\mathfrak{E}_p\circ\mathfrak{E}_q$ ( $r^{-1}=p^{-1}+q^{-1}$) for Approximation and Entropy ideals.

For Approximation ideals there is no difficulty in extending the formula above to Lorentz spaces obtaining: $\mathfrak{A}_{pq}=\mathfrak{A}_{p_0q_0}\circ\mathfrak{A}_{p_1q_1}$ when $p^{-1}=p_0^{-1}+p_1^{-1}$ and $q^{-1}=q_0^{-1}+q_1^{-1}$. However for Entropy ideals we only obtain:

$$\mathfrak{E}_{pq}=\mathfrak{E}_{p_0,q/\theta}\circ\mathfrak{E}_{p_1,q/1-\theta}\quad,\tau=p/p_0$$

This is what we called in [C] a *proportional factorization*. It is clear that not all factorizations are proportional.

Wheter or not the general factorization formula:

$$\mathfrak{E}_{pq}=\mathfrak{E}_{p_0q_0}\circ\mathfrak{E}_{p_1q_1}\quad,\qquad p^{-1}=p_0^{-1}+p_1^{-1},\ q^{-1}=q_0^{-1}+q_1^{-1},$$

holds, it is still unknown.

In this paper we extend the proportional factorization formula to Lorentz-Marcinckiewicz entropy ideals $\mathfrak{E}_{\varphi,q}$ ,$\varphi$ a suitable function and $0<q\leq+\infty$. The first point is to establish the meaning of the word "propotional" for generalized entropy ideals: In the classical case, the initial function is $\varphi(t)=t^{1/p}$, and the admissible factorizations are $\varphi=\varphi_0\cdot\varphi_1$ with $\varphi_i(t)=t^{1/p_i}$,i=0,1, and $p^{-1}=p_0^{-1}+p_1^{-1}$. The value $\tau$ is obtained as $p/p_0$, and the adequate space to factorize the initial operator is an intermediate space of K and J types $\chi(t)=t^{\theta}$. The connection between $\tau$ and those functions is:

$$\tau=\alpha_{\overline{\chi}}=\beta_{\overline{\chi}}=\frac{\alpha_{\overline{\varphi_0}}}{\alpha_{\overline{\varphi}}}=\frac{\beta_{\overline{\varphi_0}}}{\beta_{\overline{\varphi}}}$$

where $\alpha$ and $\beta$ are the so-called Boyd indices (see [G]).

In accordance with this we shall impose the restriction $\alpha_{\overline{\varphi}}=\beta_{\overline{\varphi}}$ to the initial function, and if $\varphi=\varphi_0\cdot\varphi_1$ is an admissible factorization of $\varphi$ then $\alpha_{\overline{\varphi_0}}=\beta_{\overline{\varphi_0}}$ .

Now, a simple comparison (see also the diagram at the the end of this note) gives that the intermediate function has to be $\chi=\varphi_0\circ\varphi^{-1}$. Under the preceeding hypotheses it can be proved that $\alpha_{\bar{\chi}} = \beta_{\bar{\chi}} = \frac{\alpha_{\bar{\varphi}_0}}{\alpha_{\bar{\varphi}}}$. We call $\tau$ to this value. We have:

**Theorem 1. Assume the preceeding hypotheses but $q<+\infty$ and $0<\alpha_{\bar{\varphi}_0}<\alpha_{\bar{\varphi}}$. Then if $T\in\mathfrak{E}_{\varphi,q+\varepsilon}$ for all $\varepsilon>0$ there exist $T_0\in\mathfrak{E}_{\varphi_0,(q/\tau)+\varepsilon}$ for all $\varepsilon>0$ and $T_1\in\mathfrak{E}_{\varphi_1,(q/1-\tau)+\varepsilon}$ for all $\varepsilon>0$ such that $T=T_1\circ T_0$.**

Obviously, when we let $q=+\infty$ most of the difficulties disappear and we have:

**Theorem 2. Assume $q=+\infty$. Then if $0<\alpha_{\bar{\varphi}_0}<\beta_{\bar{\varphi}}$, $\mathfrak{E}_{\varphi,\infty}=\mathfrak{E}_{\varphi_0,\infty}\circ\mathfrak{E}_{\varphi_1,\infty}$.**

Note that in this case "proportional" is meaningless and thus we do not need to ask $\alpha_{\bar{\varphi}}=\beta_{\bar{\varphi}}$ nor $\alpha_{\bar{\varphi}_0}=\beta_{\bar{\varphi}_0}$.

The behaviour of the entropy numbers under real interpolation with functional parameter is calculated in order to provide Th1 and Th2:

Assume that $T:X_0 \longrightarrow X_1$ is an interpolation couple and X is an intermediate space. We call $T_0$ and $T_1$ to the induced operators $X \longrightarrow X$ and $X \longrightarrow X_1$. We have:

| | $T \in \mathfrak{E}_{\varphi,\infty}$ | $T \in \mathfrak{E}_{\varphi,q}$ |
|---|---|---|
| If X is of K-type $\chi$ | $T_1\in \mathfrak{E}_{\rho,\infty}$ where $\rho=\varphi/\chi(\varphi)$ | $T_1 \in \mathfrak{E}_{\rho,q/1-\tau}$ where $\rho=\varphi/\chi(\varphi)$ and $\alpha_{\bar{\varphi}}<\tau<1$ |
| If X is of J-type $\chi$ | $T_0\in \mathfrak{E}_{\rho,\infty}$ where. $\rho=\chi(\varphi)$ | $T_0\in \mathfrak{E}_{\rho,q/\tau}$ where $\rho=\chi(\varphi)$ and $0<\tau<\beta_{\bar{\varphi}}$ |

## References

[C] Jesús M.F.Castillo. Factorization of Entropy Ideals: Proportional case. Portugaliae Mathematica (to appear)

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