Approximation of Convex Bodies by Polynomial Bodies III: Area Case

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With this paper we conclude one part of a work, initiated in [3] and
[4], concerning the approximation of planar symmetric convex bodies by poly-
nomial bodies. We use the notations of [3] and [4], and we refer the reader
to these papers whenever definitions are omitted, and to see the basic sche-
me of the problem. (See the above extracta of the author in this journal).

1.- We can read in Day [1](1947): "Loewner has shown (in some unpub-
lished work) that there exists a unique ellipse E of minimal area circumscri-
bed about S, and that this ellipse touches S in at least four points...". We
have generalized this result in [3] and [4] for the width and radius cases,
and we now do this for the area case. First, a simple consequence of the
strict convexity of the function \( D_{\lambda}(P) = m[B_{P}] \), proves the

**Theorem (1).** There exists a unique \( P \in P_{\gamma}(B) \) such that \( m[B_{P}] = m[B_{0}] \), \( Q \in P_{\gamma}(B) \).

**Theorem (2).** Let \( B_{P} \in B_{n}^{(1)}(B) \) \( B_{P} \in B_{n}^{(1)}(B) \). Then \( S_{P} \cap S \) has at least \( 2k+2 \) points.

Now, for the area criterion we have a complementary result of (2), not true for the width and radius criteria. Let \( B_{P} \in B_{n}^{(1)}(B) \) and \( C = \left| S_{P} \cap S \right| \), the convex envelope of \( S_{P} \cap S \). We have that \( C \in C_{n} \) and \( P \in P_{\gamma}(C) \), because \( C \subset B_{P} \), and

**Theorem (3).** \( B_{P} \in B_{n}^{(1)}(C) \).

**Corollary (4).** Let \( E \) be the ellipse of Loewner of minimal area circumscribed
about \( B \). If \( E \cap S = \{x, y, -x, -y\} \), then \( x \) and \( y \) are mutually Birkhoff orthogonal.

2.- Now we define the best approximation area-exterior operator. For
each \( B_{P} \in B_{n}^{(1)} \), there exists a unique \( P \in P_{\gamma}(B) \) such that \( B_{P} \in B_{n}^{(1)}(B) \). So we can
define the operator \( A_{n} : B_{n} \rightarrow P_{2k} \), such that \( A_{n}(B) = P \), and \( B_{P} \in B_{n}^{(1)}(B) \).

**Theorem (5).** Let \( C_{n} \in C_{n} \), with \( n \in \mathbb{N} \). If \( n \in \mathbb{N} \), then: a) \( A_{n}(C_{n}) = P_{n} \), b) \( m[B_{P}] = m[B_{0}] \).
3.-We ask now whether the contact and draw-back theorems given by us for the three approximation criteria are the best possible. In this sense we see that, for all k∈N, there exist convex bodies Be∈∈a such that the sphere of the best approximation polynomial body -between polynomials PeP_{2k}(B)- touches the sphere S of B in 2k+2 points exactly (for the three criteria), and moves away from S in 2k+2 points too (for the width and radius cases).

THEOREM (6). Let Be∈∈a be a regular 2m-gon in R^2 centered at the origin. If PeP_{2k}(B), with 1<k<m-1, and B_p is the exterior best approximation -between polynomials of degree 2k-, with the width, radius or area criteria, then S_p is the circumscribed circumference to B.

With this result we can give a negative answer (for the best approximation of sets in ∈a by polynomial bodies) of a pretty result due to Dowker [2] (1944) that established the following: "If M_n means the n-gon of minimum area circumscribed around a convex region R in the plane, then the area of M_n is a convex function of n". This is to say 2m[M_n]≤m[M_{n-1}]+m[M_{n+1}]. In our case we raise a natural question: if PeP_{2k}(B) and B_p ∈∈B(B) -between polynomials of degree 2n-, does the following inequality hold: 2m[B_n]≤m[B_{n-1}]+m[B_{n+1}]? The answer is negative.

5.-Although for width and radius cases, the unicity of best approximation was a simple consequence of the contact and draw-back theorems, in the area-interior case the situation is different. We don't know whether or not there exists a proof for it, even though we suspect that there is. We must content ourselves with a (apparently) weak result, although we will have, as a consequence, the continuity theorem if the unicity of best approximation were proved.

THEOREM (7). Let C,C_n ∈∈∈a, with n∈N. If B_p ∈∈B_n(C_n), for each n∈N, and C_n→C, then: a) P_n is bounded (then (P_n)→φ), b) if Pe(P_n), then B_p ∈∈B_n(C), c) if B_p ∈∈B_n(C), then lim m[B_n]=m[B_p].

6.-We have the unicity of best approximation area-interior, of a set Be∈∈a, when we use polynomials PeP_{2k}(B) -it is the simple case of an ellipse of Loewner-. But we don't know whether this result is true for all n∈N, as we want. Nevertheless we can give a new approach to this question with a result in the way that authors as [6] Kenderov(1980), and [5] Gruber & Kenderov (1982) introduced in the problem of the approximation of convex bodies by
polygons. In this problem, the unicity of best approximation area interior (or exterior), by \( n \)-gons, is trivially not true in general. But they observed that the convex sets, in which the unicity is not true, are few in relation to all the convex sets. Because they form a set of first category with the locally compact topology of the Hausdorff metric in \( \mathfrak{E}_a \).

THEOREM (8). \( \mathfrak{E}_{m} = (C \in \mathfrak{E}_a : 3B_{P}, R \in \mathfrak{B}_{2k}(C); P, Q \in \mathcal{P}_{2k}(C), m[B_{P} \cap B_{Q}] \geq 1/n) \) are closed with empty interior. Then \( \mathfrak{E}_{n} = \bigcap_{k=1}^{n} \mathfrak{E}_{nk} \) is dense in \( \mathfrak{E}_a \), and for all \( B \in \mathfrak{E}_a \) we have the unicity of best approximation area-interior for all \( k \in \mathbb{N} \).

7. We will see that for the interior approximation we have a "dual" result of (3), which we call the "caged amoeba" theorem, because loosely speaking, it says: "The greatest amoeba (polynomial body), of \( 2k \) pseudopodia (of degree \( 2k \)), confined into an oval enclosure, cannot fatten any more however much she tries", even if we increased the enclosure by placing straight walls at the spots where the amoeba rested before.

THEOREM (9). Let \( \kappa \in \mathbb{N} \) such that \( \bigcup \mathfrak{E}_{nk} : n \in \mathbb{N} = \emptyset \). If \( B \in \mathfrak{E}_a \), \( B_{k} \in \mathfrak{B}_{2k}(B) \), and \( \mathfrak{B}_{k} \in \mathfrak{B}_{2k}(B) \) is the convex set defined by the tangents to \( B \) at the points of \( S_{P} \), then \( B_{k} \in \mathfrak{B}_{2k}(B) \).

COROLLARY (10). If \( E \) is the ellipse of Loewner of maximal area inscribed into \( B \in \mathfrak{E}_a \) and \( E \cap B = \{(x, y, -x, -y) \} \), then \( x \) and \( y \) are Birkhoff orthogonal.

REFERENCES

[4]. FARO RIVAS,R.- "Approximation of symmetric convex bodies by polynomial bodies II. Unicity and contact theorems in width and radius cases". To appear in J.Approx.Theory.

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