We consider the equation (see [3]):

(1) \[ u(x) = \int_0^x k(x-s)g(u(s))ds \quad (x>0) \]

where

(k) \[ k: [0, +\infty) \rightarrow [0, +\infty) \]

is such that \( \int_0^x k(s)ds > 0 \) for \( x>0 \).

and

(g) \[ g \] is a continuous nondecreasing function such that \( g(0) = 0 \), \( g(u) > 0 \) for \( u > 0 \) and \( g(u)/u \rightarrow +\infty \) as \( n \rightarrow 0^+ \).

Let us note, that \( u = 0 \) is the trivial solution to (1). But we consider continuous solutions \( u \) such that \( u(x) > 0 \) for \( x > 0 \). These are so-called nontrivial solutions. We ask about the uniqueness and the possibility of the extension to the maximal interval of nontrivial solutions.

If we assume additionally

\( g_{1}(u) \) \[ g(u)/u \] is strictly decreasing and \( g(u)/u \rightarrow 0 \) as \( n \rightarrow +\infty \).

Then we can show two following theorems ([1], [2]).

**Theorem 1.** Assume (k), (g) and \( g_{1}(u) \) are satisfied. If equation (1) has a nontrivial solution on \( [0, x_0] \), \( (x_0 > 0) \), then it is unique and nondecreasing on \( [0, x_0] \).

**Theorem 2.** Assume (k), (g) and \( g_{1}(u) \) are satisfied. If equation (1) has a nontrivial solution on \( [0, x_0] \), \( (x_0 > 0) \), then it can be extended to the unique nontrivial solution on \( [0, +\infty) \).

But we can prove Theorem 1 under weaker assumptions:

**Theorem 1'.** Assume (k) and (g) are satisfied. If equation (1) has a nontrivial solution on \( [0, x_0] \), then it is unique on \( [0, x_0] \). Moreover it is nondecreasing.

We can show this theorem in two steps:
Lemma 1. If equation (1) has a nontrivial solution $u$ on $[0, x_0]$, $(x_0, 0)$, then there exists $\delta_0 > 0$ $(\delta_0 \not\in x_0)$ such that $u$ is unique on $[0, \delta_0]$. Moreover $u$ is nondecreasing on $[0, \delta_0]$.

The proof of this lemma can be found in [2].

Lemma 2. Let $u_i (i=1, 2)$ be two nontrivial solutions to (1) on $[0, x_0]$, $(x_0, 0)$ such that $u_i(x) = u_i(0)$ on $[0, \delta_0]$, $(\delta_0 \not\in (0, x_0))$. Then $u_1 = u_2$ on $[0, x_0]$.

Proof of Lemma 2 (for the comparison see [1]).

Let

$$M = \max \{ \max_{s \in [\delta, x_0]} \frac{(g(u_1(s))/u_1(s))}{u_1(s)} : i=1, 2 \}$$

and

$$m = \min \{ \min_{s \in [\delta, x_0]} \frac{(g(u_1(s))/u_1(s))}{u_1(s)} : i=1, 2 \}$$

We get

$$m \leq g(u_i(x)) \leq M \leq M_i(x)$$

for $x \in [\delta, x_0]$ and $i=1, 2$. Hence

$$||g(u_1(x)) - g(u_2(x))|| \leq (M-m) |u_1(x) - u_2(x)|$$

for $x \in [\delta, x_0]$. The result follows by standard methods.

Corollary 1. The nontrivial solution $u$ is nondecreasing on $[0, x_0]$.

Let $(0, \omega)$, $(\omega > 0)$ or $\omega = +\omega$ be the maximal interval of existence of the nontrivial solution $u$. By Theorem 1 it is easy to see that $u$ is the unique nontrivial solution to (1) on $[0, \omega)$. Moreover it is nondecreasing.

Corollary 2. If $[0, \omega)$ is the maximal interval of the existence of the nontrivial solution $u$, then either $\omega < \omega$ and $\lim_{x \to \omega^-} u(x) = +\omega$ or $\omega = +\omega$.

Now we present an example. We consider equation (1) with $k(x) = 2$ and

$$g(\cdot) = \begin{cases} u^2_2 & \text{for } u_2 \in [0, 1] \\ u^2 & \text{for } u_1 \end{cases}$$
We can easily compute that
\[ u(x) = \begin{cases} 
    x^2 & \text{for } x \in [0, 1] \\
    1/(3-2x) & \text{for } x > 1
\end{cases} \]
is the unique nontrivial solution to (1) with the maximal interval equal to \([0, 3/2]\).

We can formulate:

**Theorem 2'.** Assume \((k)\) and \((g)\) are satisfied. If equation (1) has a nontrivial solution on \([0, x_0]\) and for \(x_1 > x_0\) is
\[ K(x_1) \leq \sup_{u \in [0, \infty)} \left\{ \frac{\partial}{\partial x} (u + g(u)) \right\} \]
then the solution \(u\) can be extended to \([0, x_1]\).

**Corollary 2.** If \( \lim_{x \to \infty} K(x) \leq \sup_{u \in [0, \infty)} \left( \frac{u + g(u)}{2} \right) \) then every nontrivial solution to (1) (if there exists) can be extended to \([0, +\infty)\).

For similar proofs look [2].

**REFERENCES**

