NONTRIVIAL SOLUTIONS TO NONLINEAR VOLterra EQUATIONS

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AMS (1980) Class Number: 45D05, 45G10

In some physical problems the following nonlinear Volterra equations

\[ u(x) = \int_0^x k(x-s)g(u(s))ds \]

where

\[ k: (0, \delta_0) \rightarrow (0, +\infty), \quad (\delta_0 > 0) \]

is an absolutely continuous monotonous function such that \( \int_0^\infty k(s)ds < +\infty \) and

\[ g: (0, +\infty) \rightarrow (0, +\infty) \]

is an absolutely continuous increasing function such that \( g(0) = 0 \) and \( g(x) > 0 \) for \( x > 0 \),

are considered (see [4]). With respect to physical applications the continuous solutions \( u \) to (1) such that \( u(x) > 0 \) for \( x > 0 \) are interesting. These are so-called nontrivial solutions.

On the base of Gripenberg’s work (see [3]) we can formulate:

Theorem 1. Let \( \kappa > 0 \). Assume:

i) \( g(u)/u \) is continuous positive decreasing such that \( g(u)/u \rightarrow +\infty \) as \( u \rightarrow 0^+ \).

ii) For each \( q > 0 \) the function \( u^{[g(u)/u]^q} \) is increasing on \( [0, \delta_q] \), \( (\delta_q > 0) \). The equation

\[ u(x) = \int_0^x (x-s)^{q-1}g(u(s))ds \]

has a nontrivial solution on \( [0, \delta] \), \( (\delta > 0) \) if and only if

\[ \int_0^\delta \left[ u^{1/\kappa} \right]^{1/\kappa} \frac{du}{u} < +\infty \]

But we must emphasize that under Gripenberg’s assumptions the case \( g(u) = u^p \) (\( p \in (0, 1) \)) is not allowed. In papers [1], [2], and [5] under weaker assumptions of \( g \) similar results to Theorem 1 are presented. At these works the case \( g(u) = u^p \) (\( p \in (0, 1) \)) is allowed. Here we want to present generalizations of condition (3). In all Theorems below \( K^{-1} \) will
denote the inverse function to $K(x) = \int_0^x k(s) ds$. Now we formulate two theorems with sufficient conditions for the existence of nontrivial solutions to (1).

**Theorem 2.** Let $k$ be an increasing function satisfying (k) and $g$ satisfy (g). If
\[
(4) \quad \int_0^\delta \frac{g'(u)}{g(u)} \frac{k^{-1}(u/g(u))}{u} du < \infty
\]
then equation (1) has a nontrivial solution on $[0, \delta] (\delta > 0)$.

**Corollary 1.** If $g$ satisfies additionally (i) then $ug'(u)g(u)$. Suppose
\[
(5) \quad \int_0^\delta \left( \frac{k^{-1}(u/g(u))}{u} \right) du < \infty
\]
Then (4) is satisfied and by Theorem 2 equation (1) has a nontrivial solution.

**Theorem 3.** Let $k$ be a decreasing function satisfying (k) such that $\log k$ is convex and $g$ satisfies (g). If
\[
(6) \quad \int_0^\delta \frac{g(u)k \cdot k^{-1}(u/g(u))}{u} du < \infty
\]
Then equation (1) has a nontrivial solution on $[0, \delta] (\delta > 0)$.

Now we give two theorems concerning necessary conditions.

**Theorem 4.** Let $k$ be an increasing function satisfying (k) such that $\log k$ is concave and $g$ satisfies (g). If equation (1) has a nontrivial solution then
\[
(7) \quad \int_0^\delta \left( \frac{g(u)k \cdot k^{-1}(u/g(u))}{u} \right) du < \infty
\]

**Theorem 5.** Let $k$ be a decreasing function satisfying (k) and $g$ satisfies (g). If equation (1) has a nontrivial solution then
\[
(8) \quad \int_0^\delta \frac{g'(u)/g(u)}{k^{-1}(u/g(u))} du < \infty
\]

**Corollary 2.** If there exists $q_o > 1$ such that $u[g(u)/u]^{q_o}$ is increasing then $g'(u) > (1-1/q_o) g(u)/u (|3|)$. In this case if equation (1) has a nontrivial solution then
\[
(9) \quad \int_0^\delta \frac{k^{-1}(u/g(u))}{u} du < \infty
\]
Remark 1. Let us note that in the case of $k(x) = x^{k-1}$ the conditions (5), (6), (7) and (9) are equivalent to Gripenberg's condition (3). By Corollary 1, Theorems 3 and 4 and Corollary 2 we get Theorem 1. Moreover all these conditions work in the case of $g(u) = u^p (p \in (0,1))$.

Remark 2. It is known that equation (1) has a nontrivial solution in the case of $k(x) = \exp(-1/x^p)$ ($p \geq 1$) and $g(u) = u^p (p \in (0,1))$. But our sufficient conditions do not work in this case.

REFERENCES.


