GAP AND PERTURBATION OF NON-SEMI-FREDHOLM OPERATORS

Jose A. Alvarez*, Teresa Alvarez*, Manuel Gonzalez*

*Departamento de Matematicas, Universidad de Cantabria, Spain
+Departamento de Matematicas, Universidad de Oviedo, Spain

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A classic result of Fredholm theory states that perturbing a semi-Fredholm operator $T$ (i.e., an operator with closed range $R(T)$ and finite dimensional kernel $N(T)$ or cokernel $Y/R(T)$) by a continuous operator $A$ with norm smaller than the minimum modulus $\gamma(T)$ of $T$ (or by a compact operator $A$) we obtain a semi-Fredholm operator $T+A$; moreover, $\dim N(T+A) \leq \dim N(T)$ and $\text{codim } R(T+A) \leq \text{codim } R(T)$ (see [3]).

Summing up, for operators with closed range the finite dimension of the kernel and the cokernel are stable under small or compact perturbations.

However, for any operator with non-closed range we can find a compact operator with arbitrarily small norm such that the perturbed operator has infinite dimensional kernel and cokernel [4].

In [1] Bouldin divides the class NSF of all bounded non-semi-Fredholm operators in a separable Hilbert space, in six disjoint sets $U_j$ $0 \leq i \leq 5$. He proves that for $0 \leq j \leq 4$ the sets are totally unstable: $U_j + \text{Co} = \text{NSF}$. However, the class $U_5$ of all operators with closed range and infinite dimensional kernel and cokernel is not totally unstable.

Later, Gonzalez and Onieva [5] studied the problem for bounded operators in Banach spaces, obtaining the same results for $U_j$ $0 \leq j \leq 4$. These results were extended by Labuschagne [7] to closed operators. However, in [5] and [7] the stability of $U_5$ is not investigated.

In this paper we shall prove that certain isomorphic properties of Banach spaces (separability, reflexivity, containing no copies of $\ell_1$, ...) for the kernel or the cokernel of a closed operator with closed range are stable under small or compact perturbations. As an application we prove that some subsets of $U_5$ are not totally unstable under compact perturbation.

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In order to do that, we use several known results about the stability of some properties of the subspaces of a Banach space under small perturbations measured by the gap (see [6, 9]). We also obtain here some new results of this kind and give new proofs for some of the known ones. For example, it will be shown that given a property $P$ which is stable under the gap, the property $P^{co}$ ($X \in P^{co} \iff X^{**} \cap J(X) \in P$) is also stable. From here we derive the stability of the reflexivity [2, Th. 5] and the quasi-reflexivity as a consequence of the stability of the finite dimension.

Given two closed subspaces $M, N$ of $X$, $M^{\perp}$ and $\delta(M,N)$ will be the annihilator of $M$ and the gap between $M$ and $N$, defined by

$$\delta(M,N) := \sup \{ \text{dist}(m,N) : m \in M, \|m\| = 1 \}.$$ 

The minimum modulus is $\gamma(T) := \inf \{ \|Tx\| : \text{dist}(x,N(T)) = 1 \}$

Theorem (Markus [8]) Let $T$ be a closed operator with $\gamma(T) > 0$, $A$ a continuous operator and $S = T+A$.

(a) $\delta(\mathcal{N}(S),N(T)) \leq \gamma(T)^{-1}\delta(S,T)$.  
(b) $\delta(\mathcal{R}(T),R(S)) \leq \gamma(T)^{-1}\delta(S,T)$.

Definition 1 A property $P$ will be a class of Banach spaces stable under isomorphisms. A Banach space $X$ has property $P$ when $X \in P$.

Let $0 < a \leq 1$. $P$ is $a$-stable ($a$-costable) if for every Banach space $X$ and closed subspaces $M, N$ of $X$ we have:

$N \in P, \delta(M,N) < a \Rightarrow M \in P$. ($X/M \in P, \delta(M,N) < a \Rightarrow X/N \in P$).

$P$ is stable (costable) if it is $a$-stable ($a$-costable) for some $a$.

Observation 2 (a) The dual property $P^{d} := \{ X : \text{the dual } X^{*} \in P \}$ is $a$-costable ($a$-stable) when the property $P$ is $a$-stable ($a$-costable).

(b) Finite dimension is a 1-stable and 1-costable property.

Theorem 3 (a) The classes of superreflexive, containing no copies of $\ell_1$ and B-convex Banach spaces are 1-stable.

(b) The class of superreflexive Banach spaces is 1-costable.

(c) The class of separable spaces is 1/2-stable and 1/2-costable.

Theorem 4 If $P$ is an $a$-stable ($a$-costable) property then the property $P^{co} := \{ X : H(X) \in P \}$ is $a/2$-stable ($a/2$-costable).

In particular, the quasi-reflexivity and the reflexivity are 1/2-stable and 1/2-costable; and $\{ X : H(X) \text{ is separable} \}$, studied in [10] is 1/4-stable and 1/4-costable.
Proposition 5  (a) If $P$ is a-costable then $P$ is a/2-stable.
(b) If $P$ is a-stable then $P$ is a/2-costable.
(c) The classes of superreflexive, containing no copies of $\ell_1$ and B-convex Banach spaces are i/2-costable.

Theorem 6  Let $P$ be a property, $T$ a closed operator with $\gamma(T) > 0$, $0 < a \leq 1$ and $A$ a continuous operator such that $\|A\| < a\gamma(T)$.
(a) If $P$ is a-stable and $N(T) \in P$, then $N(T+A) \in P$.
(b) If $P$ is a-costable and $Y/R(T) \in P$, then $Y/R(T+A) \in P$.

Theorem 7  Suppose $X\times N \in P \Leftrightarrow X \in P$, when $\dim N < \omega$. Let $T$ be a closed operator with $\gamma(T) > 0$, $K$ a compact operator, and $0 < a \leq 1$.
(a) If $P$ is a-stable and $N(T) \in P$, then $N(T+K) \in P$.
(b) If $P$ is a-costable and $Y/R(T) \in P$, then $Y/R(T+K) \in P$.

Proposition 8  Let $P$ be a property such that $X\times Y \in P$ when $X, Y \in P$. If $P$ verifies the result in 8.a or 8.b, then $P$ has the three space property; i.e., $X \in P$ when it has a subspace $M$ such that $M, X/M \in P$.

REFERENCES