DUALITY PROPERTIES OF INJECTIVE MODULES

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A Morita duality between two rings $R$ and $T$ is always induced by a faithfully balanced bimodule $R_U^T$ such that $R_U$ and $U_T^R$ are injective cogenerators. In [13], an asymmetrical generalization of Morita duality has been given by considering "duality $R$-modules", i.e., bimodules $R_U^T$ such that $R_U$ is a finitely cogenerated linearly compact quasi-injective self-cogenerator and $T$ is naturally isomorphic to $\text{End}_R(U)$. It has been remarked in [13] that these modules can be regarded as "$\Omega$-Morita duality modules", because $R_U^T$ is a Morita duality module if and only if it is a duality $R$-module and a (right) duality $S$-module. However, they are in some sense rather more than "$\Omega$-Morita duality modules" for if $R_U$ is a module and $T = \text{End}_R(U)$, then $U_T$ can be an injective cogenerator without $R_U$ being finitely cogenerated nor linearly compact nor a self-cogenerator. Thus the question arises of giving necessary and sufficient conditions on $R_U$ for $U_T$ to be an injective cogenerator.

We will attack this problem by looking first at the simpler one of determining when $U_T$ is injective, i.e., when $R_U$ is a counterinjective module. There is a result due to Würfel [12] and Damiano [4] which will be helpful for this purpose, namely, $U_T$ is FP-injective (i.e., $\text{Ext}_T(F,U) = 0$ for every finitely presented right $T$-module $F$) if and only if $R_U$ cogenerates all cokernels of homomorphisms $U^m \rightarrow U^n$. It is also well known that the linear compactness of $R_U$ is closely related to the injectivity of $U_T$ (see [8], [10]). But it is easily seen that a counterinjective module is not necessarily linearly compact (in the discrete topology) and so we will make use of the following more general concept: A left $R$-module $X$ will be called $U$-linearly compact when each finitely solvable system of congruences $x \equiv x_i (\text{mod } X_i)$, with the $X_i$ $U$-closed submodules of $X$ (i.e., such that $X/X_i$ is $U$-cogenerated), is solvable [6]. We then get the following characterization of counterinjective modules:

Theorem 1. A left $R$-module $R_U$ is counterinjective if and only if the following conditions hold:

1. $R_U$ is finitely cogenerated,
2. $R_U$ is linearly compact,
3. $R_U$ is linearly $U$-compact,
4. $R_U$ cogenerates all cokernels of homomorphisms $R_U \rightarrow R_U^n$.
i) Every cokernel of a homomorphism of the form $U^m \longrightarrow U^n$ is $U$-cogenerated.

ii) $U$ is $U$-linearly compact.

Several results scattered in the literature can be recovered as easy corollaries of Theorem 1. For instance we mention [8, Coroll. 1, p. 119], [10, Coroll. 2, p. 342], and [9 Theorem 1]. As another application, we get the following characterization of rings with Morita duality which improves [6, Corollary 3].

**Corollary 2.** Let $R^T_U$ be a faithfully balanced bimodule such that $R^U_U$ is a cogenerator and the injective envelope of $U^T_T$ is cogenerated by $U^T_T$. Then $R$ has a left Morita duality.

The proof consists in using results of [3] and [7] to show that $R^U_U$ is counterinjective and then one can apply Theorem 1.

If we consider faithfully balanced bimodules $R^U_U$ that are injective and counterinjective, it is easy to see that they induce a Morita duality (in the sense of [11]) between the quotient categories of $R$-Mod and $R$-Mod modulo the localizing subcategories defined by $R^U_U$ and $U^T_T$. The following result gives an indication of how close is $R$ to having a Morita duality in this case. Recall that an $R$-module $X$ is said to have $U$-dominant dimension $\geq 2$ ($U$-dom dim $X \geq 2$) if there exists an exact sequence $0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_2$, in which $X_1$ and $X_2$ are direct products of copies of $R^U_U$.

**Theorem 3.** Let $R^U_U$ be an injective and counterinjective module. If every direct sum of copies of $R^U_U$ has $U$-dom dim $\geq 2$, then $T$ is semiperfect. If $U$ is faithfully balanced and the class of modules of $U$-dom dim $= 2$ is closed under direct unions, then $R$ has a left Morita duality.

As a consequence we get the following extension of [5, Coroll. 10.14]. Recall that an $R$-module $X$ is called $\Sigma$-injective (resp. $\Delta$-injective) when it is injective and $R$ satisfies the ascending (resp. descending) chain condition on annihilators of subsets of $X$.

**Corollary 4.** Let $R^T_U$ be a faithfully balanced bimodule such that $R^U_U$ is $\Sigma$-injective (resp. $\Delta$-injective) and $U^T_T$ is injective. Then $R$ is a left noetherian (resp. artinian) ring with a left Morita duality.

Recall that a module $U$ is quasi-injective if, for every submodule $X$ of $U$, the canonical homomorphism $\text{Hom}_R(U,U) \longrightarrow \text{Hom}_R(X,U)$ is an epimorphism. Using Theorem 1 we can now characterize the quasi-injective modules $R^U_U$ such that $U$ is an injective cogenerator.

**Theorem 5.** Let $R^U_U$ be a quasi-injective module and $T = \text{End}_R(R^U_U)$. 

Then the following conditions are equivalent:

i) $U_T$ is an injective cogenerator

ii) $U^T$ satisfies the following conditions:
   a) $U^T$ cogenerates all the cokernels of homomorphisms $U \longrightarrow U^p$.
   b) $U^T$ is $U$-linearly compact.
   c) The lattice of $U$-closed submodules of $U^T$ has the finite intersection property.

iii) Every cyclic right $T$-module and every $U$-cogenerated quotient of $U$ are $U$-reflexive.

We remark that the artinian injective modules $U^T$ which cogenerate an exact torsion theory (see [11] for the definition) satisfy all the conditions in ii) of Theorem 5, so that in this case, $U_T$ is not only injective as asserted in [2, Theorem 4.2] but is also a cogenerator.

The proofs of the foregoing results, except Corollary 2, will appear in [6].

References