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PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS WITH STRONG RESONANCE AT INFINITY.

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In this note we present some of the results of [2], where we study the existence and multiplicity of T-periodic solutions for the hamiltonian system of second order

$$-x''=\nabla_x V(t,x)+h(t) \qquad (1)$$

where $V\in C^1(\mathbb{R}x\mathbb{R}^K,\mathbb{R})$ is a T-periodic (T>0) function in the variable t and $h\in L^2([0,T],\mathbb{R}^K)$ is such that $\int_0^T h(t)dt=0$. It is assumed that (1) is strongly resonant at infinity, i.e.

(V_1) $V(t,x)\longrightarrow 0$, when $|x|\longrightarrow\infty$ (uniformly in [0,T]).

(V_2) $\nabla_x V(t,x)\longrightarrow 0$, when $|x|\longrightarrow\infty$ (uniformly in [0,T]).

Such kinds of problems have been studied by many authors. In particular in [1], [4] and [6], for $h\equiv 0$, it is proved that if V satisfies (V_1), (V_2) and moreover, one of the following assumptions:

$$\exists\ r>0 \ :\ V(t,x)<0 \quad \forall t\in[0,T],\ \forall x\in\mathbb{R}^K : |x|\geq r \qquad (2)$$

or

$$\exists\delta>0,\ \exists\zeta\in\mathbb{R}^K : V(t,x)<0\ \forall t\in[0,T],\ \forall x\in\mathbb{R}^K: |\zeta-x|\leq T\sqrt{\frac{M_1}{2}}+\delta \qquad (3)$$

(M_1= Sup $\{V(t,x)\ /\ (t,x)\in\mathbb{R}x\mathbb{R}^K\}$)

or

$$\exists x\in\mathbb{R}^K : \int_0^T V(t,x)\ dt > 0 \qquad (4)$$

then (1) has at least one weak T-periodic solution.

Moreover, in [4], if the author wants to prove existence of solution for all $h\in L^2([0,T],\mathbb{R}^K)$ with $\int_0^T h(t)dt=0$, he needs, besides (V_1) and (V_2), that $V(t,x)<0$ $\forall (t,x)\in\mathbb{R}x\mathbb{R}^K$.

Our starting point (inspired in [7]) consists in showing that the conditions (V_1), (V_2) imply that (1) has at least one weak T-periodic solution.

<u>*THEOREM 1*</u>*: Suppose V satisfies* (V_1) *and* (V_2)*. Then (1) has a weak T-periodic solution for all* $h\in L^2([0,T],\mathbb{R}^K)$ *with* $\int_0^T h(t)\,dt=0$.

To prove this, we use the variational structure of (1), corresponding the weak solutions of (1) to the critical points of the functional $I:E\longrightarrow\mathbb{R}$ defined by

$$I(u)=\frac{1}{2}\int_0^T |u'(t)|^2\,dt - \int_0^T V(t,u(t))\,dt - \int_0^T (h(t)|u(t))\,dt,$$

where $E=H^1(S^1,\mathbb{R}^K)$ is the usual Sobolev space in this kind of problems. To prove that I has a critical point we use some of the ideas of [3] and [7]. The main difficulty to apply this method is that I does not verify the compactness condition of Palais-Smale $(P\text{-}S)_d$ for all $d\in\mathbb{R}$. In fact, I verifies this condition except in a real value. The idea to overcome this problem is to consider two possible critical values of I: the infimum m of I and a min-max value c obtained by the saddle point theorem of Rabinowitz ([5]). Then we prove, (because the condition $(P\text{-}S)_d$ fails only in one value), that either m or c is a critical value of I.

In the proof of Theorem 1 we are not able to distinguish which one of these values, c or m, is the critical value of I. Because of this, we give a sufficient and necessary condition for I to attain its infimum m:

<u>*PROPOSITION 2*</u>*: If there exists* $u\in E$ *such that*

$$\frac{1}{2}\int_0^T |u'(t)|^2\,dt - \int_0^T V(t,u(t))\,dt - \int_0^T (h(t)|u(t))\,dt \le$$

$$\le \frac{-1}{2}\int_0^T |v_0'(t)|^2\,dt \qquad (5)$$

where v_0 *is any solution in E of the equation* $-x''=h(t)$*, then (1) has at least a solution* $u_0\in E$ *with* $I(u_0)=m$.

And a sufficient condition for c to be a critical value:

<u>*PROPOSITION 3*</u>*: Let* $M=\sup_{(t,x)\in[0,T]\times\mathbb{R}^K}|\nabla_x V(t,x)|$.

If there exists $\zeta\in\mathbb{R}^K$ *such that the quantity*

$$\frac{-T^2}{8\pi^2}\left[\left(\int_0^T |h(t)|^2 dt\right)^{1/2} + M\sqrt{T}\right]^2 - \int_0^T V(t,\zeta)\,dt + \frac{1}{2}\int_0^T |v_0'(t)|^2\,dt$$

is greater than zero, *(6)*

then (1) has at least one weak T-periodic solution $u_1 \in E$ with

$$I(u_1) \geq \frac{-T^2}{8\pi^2}\left[\left(\int_0^T |h(t)|^2 dt\right)^{1/2} + M\sqrt{T}\right]^2 - \int_0^T V(t,\zeta)dt > m.$$

REMARKS 4: In [6] and for $h \equiv 0$, Thews proved that I attains its infimum if (in addition to (V_1) and (V_2)) (4) is satisfied. Taking a constant function in (5) we see that a nonstrict inequality in (4) is sufficient for it. Also, a different condition from (6) may be given, which generalizes (3) and implies that c is a critical value of I, greater than m, in this case.

As a consequence of this, we obtain sufficient conditions for the existence of at least two solutions which correspond to the distinct critical values of I: c and m.

COROLLARY 5: If V satisfies all conditions of Propositions 2 and 3, then (1) has at least two weak T-periodic solutions in E which correspond to different critical levels of I.

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