PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS
WITH STRONG RESONANCE AT INFINITY.

D. Arcoya

Departamento de Análisis Matemático. Universidad de Granada. 18071, Granada, Spain.
AMS Subject Classification (1980): 34C25, 34B15, 58E05.

In this note we present some of the results of [2], where we study the existence and multiplicity of T-periodic solutions for the Hamiltonian system of second order

\[-x'' = \nabla_x V(t, x) + h(t)\]

(1)

where \( V \in C^1(\mathbb{R} \times \mathbb{R}^k, \mathbb{R}) \) is a T-periodic (T>0) function in the variable t and \( h \in L^2([0,T], \mathbb{R}^k) \) is such that \( \int_0^T h(t) \, dt = 0 \). It is assumed that (1) is strongly resonant at infinity, i.e.

(\( V_1 \)) \( V(t, x) \to 0 \), when \( |x| \to \infty \) (uniformly in \([0,T]\)).

(\( V_2 \)) \( \nabla_x V(t, x) \to 0 \), when \( |x| \to \infty \) (uniformly in \([0,T]\)).

Such kinds of problems have been studied by many authors. In particular in [1], [4] and [6], for \( h=0 \), it is proved that if \( V \) satisfies (\( V_1 \)), (\( V_2 \)) and moreover, one of the following assumptions:

\[ \exists \, r>0 \quad : \quad V(t, x) < 0 \quad \forall t \in [0,T], \quad \forall x \in \mathbb{R}^k \quad : \quad |x|^r > r \]  \hspace{1cm} (2)

or

\[ \exists \, \delta > 0, \exists \xi \in \mathbb{R}^k : V(t, x) < 0 \quad \forall t \in [0,T], \quad \forall x \in \mathbb{R}^k : |x - \xi| < T \left( \frac{\bar{M}_1}{2} \right)^{1/2} + \delta \]  \hspace{1cm} (3)

(\( \bar{M}_1 = \sup \{ V(x, t) / (x, t) \in \mathbb{R} \times \mathbb{R}^k \} \))

or

\[ \exists \, \delta > 0 : \int_0^T V(t, x) \, dt > 0 \]  \hspace{1cm} (4)

then (1) has at least one weak T-periodic solution.

Moreover, in [4], if the author wants to prove existence of solution for all \( h \in L^2([0,T], \mathbb{R}^k) \) with \( \int_0^T h(t) \, dt = 0 \), he needs, besides (\( V_1 \)) and (\( V_2 \)), that \( V(t, x) < 0 \quad \forall x \in \mathbb{R}^k \).

Our starting point (inspired in [7]) consists in showing that the conditions (\( V_1 \)), (\( V_2 \)) imply that (1) has at least one weak T-periodic solution.
THEOREM 1: Suppose $V$ satisfies $(V_1)$ and $(V_2)$. Then (1) has a weak $T$-periodic solution for all $h \in L^2([0,T], \mathbb{R}^k)$ with $\int_0^T h(t) \, dt = 0$.

To prove this, we use the variational structure of (1), corresponding the weak solutions of (1) to the critical points of the functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2} \int_0^T |u'(t)|^2 \, dt - \int_0^T V(t, u(t)) \, dt - \int_0^T (h(t) |u(t)|) \, dt,$$

where $E = H^1(S^1, \mathbb{R}^k)$ is the usual Sobolev space in this kind of problems. To prove that I has a critical point we use some of the ideas of [3] and [7]. The main difficulty to apply this method is that I does not verify the compactness condition of Palais-Smale $(P-S)_d$ for all $d \in \mathbb{R}$. In fact, I verifies this condition except in a real value. The idea to overcome this problem is to consider two possible critical values of I: the infimum $m$ of I and a min-max value $c$ obtained by the saddle point theorem of Rabinowitz ([5]). Then we prove, (because the condition $(P-S)_d$ fails only in one value), that either $m$ or $c$ is a critical value of I.

In the proof of Theorem 1 we are not able to distinguish which one of these values, $c$ or $m$, is the critical value of I. Because of this, we give a sufficient and necessary condition for I to attain its infimum $m$: 

**PROPOSITION 2:** If there exists $u \in E$ such that

$$\frac{1}{2} \int_0^T |u'(t)|^2 \, dt - \int_0^T V(t, u(t)) \, dt - \int_0^T (h(t) |u(t)|) \, dt \leq \frac{-1}{2} \int_0^T |v'_0(t)|^2 \, dt$$

(5)

where $v_0$ is any solution in $E$ of the equation $-x'' = h(t)$, then (1) has at least a solution $u_0 \in E$ with $I(u_0) = m$.

And a sufficient condition for $c$ to be a critical value:

**PROPOSITION 3:** Let $M = \sup_{(t,x) \in [0,T] \times \mathbb{R}^k} |V(t,x)|$.

If there exists $c \in \mathbb{R}$ such that the quantity

$$\frac{-\pi^2}{8T^2} \left[ \left( \int_0^T |h(t)|^2 \, dt \right)^{1/2} + \frac{M}{T} \int_0^T |V(t, \zeta)| \, dt + \frac{1}{2} \int_0^T |V'(t)|^2 \, dt \right]$$

is greater than zero,

(6)
then (1) has at least one weak $T$-periodic solution $u_1 \in \mathcal{E}$ with

$$I(u_1) \geq \frac{-T^2}{8\pi^2} \left[ \left( \int_0^T |h(t)|^2 \, dt \right)^{1/2} + \frac{M^2}{T} \right]^2 - \int_0^T V(t, \xi) \, dt > m.$$ 

**REMARKS 4:** In [6] and for $h=0$, Thews proved that $I$ attains its infimum if (in addition to $(V_1)$ and $(V_2)$) (4) is satisfied. Taking a constant function in (5) we see that a nonstrict inequality in (4) is sufficient for it. Also, a different condition from (6) may be given, which generalizes (3) and implies that $c$ is a critical value of $I$, greater than $m$, in this case.

As a consequence of this, we obtain sufficient conditions for the existence of at least two solutions which correspond to the distinct critical values of $I$: $c$ and $m$.

**COROLLARY 5:** If $V$ satisfies all conditions of Propositions 2 and 3, then (1) has at least two weak $T$-periodic solutions in $\mathcal{E}$ which correspond to different critical levels of $I$.

**REFERENCES.**


