A NOTE ON HALLEY'S METHOD

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In this paper, we study the influence that the convexity of a real function \( f \) has in Halley's method [4, 5], in order to get the solution of \( f(x) = 0 \). References [2] and [3] give global convergence theorems of this method.

For each convex function we introduce an index. This is the number of times that we need to compose the function with the logarithmic function in order to get a concave one. This concept, called degree of logarithmic convexity, provides a measure of the convexity of \( f \) at each point.

Let \( f \in C^2(\mathbb{R}) \) a convex and positive function, \( V \) a neighbourhood of \( x_0 \in \mathbb{R} \). Denote \( T[f](x) = f(x) - f(x_0) + 1 \) and define \( F_1 = \log f \), \( G_1 = T[F_1] \). By induction, if \( n \in \mathbb{N} \):
\[
F_{n+1} = \log G_n \quad \text{and} \quad G_{n+1} = T[F_{n+1}].
\]
If \( x_0 \) is not a minimum of \( f \), it is immediate that \( F_n \) is a convex function if and only if \( f(x_0) f''(x_0) [f'(x_0)]^{-2} \geq n \). It leads us to introduce the following

DEFINITION

The degree of logarithmic convexity of \( f \) at \( x_0 \) is defined by
\[
L_f(x_0) = \frac{f(x_0) f''(x_0)}{[f'(x_0)]^2}
\]

Notice that if \( x_0 \) is a minimum of \( f \) then \( F_n \) is convex for \( \varepsilon^n \) and so \( L_f(x_0) = +\infty \). We can extend formally this definition for every function \( f \in C^2(\mathbb{R}) \).
In terms of the degree of logarithmic convexity, Halley’s method consists in applying the iterative process given by

\[ x_n = F(x_{n-1}) \quad \text{with} \quad F(x) = x - \frac{f(x)}{f'(x)} \left( 2 - \frac{f''(x)}{f'(x)} \right) \]

It is known [1], that Halley’s method can be derived by applying Newton’s method to the function

\[ h(x) = \frac{f(x)}{[f'(x)]^{1/2}} \]

In what follows, we take \( f \) satisfying the following conditions

(3) \( f \in C^3([a,b]), f(a) < 0 < f(b), f'(x) > 0 \) and \( f''(x) > 0 \) for \( x \in [a,b] \)

These conditions imply that there exists one and only one root \( s \in (a,b) \) of the equation \( f(x) = 0 \). Suppose that the starting value \( x_0 \) satisfies \( a < x_0 < b \). If we study the convergence of Newton’s method for the function \( h \), by means of the degree of logarithmic convexity, we obtain a new theorem of global convergence for Halley’s method.

**Theorem**

(i) If \( L_f(x) \leq 3/2 \) in \([a,b]\) then \( \{x_n\} \), given by (2), is a decreasing sequence that converges to \( s \).

(ii) If \( L_f(x) \in (3/2,2) \) and \( L_f(x) < 1 \) in \([a,b]\), for \( x_0 \in (a + 2f(b)/f(a), 3 + 2f(b)/f(a)) \), then the sequence \( \{x_n\} \), given by (2), converges to \( s \).

**Proof**

(i) It is immediate that \( h \) has a point of inflexion at the root \( s \) since

\[ h'(x) = \frac{f(x)}{4[f'(x)]^{1/2}} L_f(x) \left[ 3 - 2L_f(x) \right] \]

Then, if \( L_f(x) \leq 3/2 \), it follows that \( h \) is a concave function in \((a,s)\) and a convex function in \((s,b)\). On the other hand, as \( h'' \) is a positive function in \((s,b)\) then \( h'(x) = f(x)^{1/2}(2-L_f(x))/2 \) is an increasing function in \((s,b)\). Besides, \( h(s) > 0 \) and therefore \( h \) is an increasing function in \((s,b)\). By applying Newton’s method to \( h \), we obtain that \( \{x_n\} \) is a decreasing sequence that
converges to $s$.

(ii) $L_f$ is a negative function in $[a,b]$ and $|L_f(x)| < 1$ if and only if $f'(x)(2 - L_f(x)) - f(x)L_f(x) > 0$. Then

$$L_f'(x) = \frac{f''(x)}{f'(x)} [1 + L_f(x)(L_f(x) - 2)]$$

Therefore, we obtain that $|L_f(x)| < 1$ if and only if

$$2 - L_f(x) > L_f(x)[1 + L_f(x)(L_f(x) - 2)]$$

Then, taking into account that $L_f(x) = 3(2,2)$ and $L_f(x) < 1$ in $[a,b]$, it follows that $|L_f(x)| < 1$ in $[a,b]$. Thus there exists $M = (0,1)$ such that $|L_f(x)| < M$ in $[a,b]$.

On the other hand, if we denote $H(x) = x - \frac{h(x)}{h'(x)}$ and $x_n = H(x_{n-1})$, it is immediate that $H(x_0) = (a,s)$ and $|x_1 - s| < M$ with $x_0 - s$ since $x_1 - s = H(x_0) - H(s) = H(\xi_0)(x_0 - s)$ for $\xi_0 = (s,x_0)$. By induction, we obtain that $H(x_{2n}) = (s,b)$ and $H(x_{2n+1}) = (a,s)$ for $n \geq 0$. Besides $|x_n - s| < M^n |x_0 - s|$ and therefore $\lim_{n} x_n = s$.

If the function $f$ is decreasing, all the previous results turn out to be valid by changing slightly the reasoning used. The condition $f(x_0) > 0$ does not affect the results when $f(x_0) < 0$ and $L_f'(x) \leq 3/2$, then $x_n$ turns to be an increasing sequence. If $f$ is concave, then by considering the respective degree of exponential concavity, we obtain analogous results.

Bibliography