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SEMIEMBEDDINGS AND SEMI-FREDHOLM OPERATORS

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Recently several papers have studied the preservation of isomorphic properties of Banach spaces by operators satisfying conditions weaker than those of the isomorphisms, such as semiembeddings, G_δ-embeddings and tauberian operators. See, for example [3], [4], [7], [9], [14], [15], [16] and [17].

On the other hand the authors [11], [12] have introduced semigroups canonically associated with some operator ideals defined in terms of sequences.

In this paper we introduce a semigroup SRN_+ containing the H_δ-embeddings of separable spaces, and the upper semi-Fredholm operators SF_+, but not the G_δ-embeddings of separable spaces; it is stable under perturbations in the operator ideal RN of all Radon-Nikodym operators, and its operators preserve the Radon-Nikodym property RNP : If there exists $T \in SRN_+(X,Y)$ and Y has the RNP, then X has the RNP property as well. Banach spaces without subspaces with the RNP are characterized in terms of the coincidence of SRN_+ and SF_+. We also study another semigroup SRN_- related with the lower semi-Fredholm operators SF_- and Asplund Banach spaces Asp in a dual way.

Recall that an operator $A \in L(L_1,X)$ is representable if there exists an X-valued Bochner-integrable function g in [0,1] such that $Af = \int fg\, d\mu$ for every $f \in L_1$.

Let K , $T \in L(X,Y)$. $K \in RN$: KA is representable for every operator A from L_1 into X.

$T \in SRN_+$: An operator A from L_1 into X is representable if (and only if) TA is representable.

PROPOSITION 2 Let K , $T \in L(X,Y)$ and $S \in L(Y,Z)$.

(a) If $T \in SRN_+$ then N(T) has the RNP.

(b) If S , $T \in SRN_+$ and $K \in RN$, then ST , $T + K \in SRN_+$.

(c) If Y has the RNP and $T \in SRN_+$, then X has the RNP.

EXAMPLES The following classes of operators are in SRN_+:

(1) Upper semi-Fredholm operators.

(2) Products and restrictions of H_δ-embeddings and F_σ-embeddings (semiembeddings under an equivalent norm) of separable Banach spaces.

(3) The operators $(I - \Lambda_T)$ in $L_1(G)$ associated with a well-founded tree on the set of integer numbers [2; Prop. 13].

We note that SRN_+ does not contain all the G_δ-embeddings of separable spaces (see [10] for a counterexample).

The following result, central in our paper, is a consequence of Lewis-Stegall characterization of Banach spaces with the RNP in terms of the factorization of representable operators through l_1, and the lifting property of the space l_1.

THEOREM 3 Let $T \in L(X,Y)$ an operator with closed range. If $N(T)$ has the RNP then $T \in SRN_+$.

COROLLARY 4 (a) $T \in SRN_+(X,Y)$ if and only if $N(T)$ has the RNP and the associated injective operator $\hat{T}$ is in SRN_+.

(b) The class RNP has the three space property [5].

THEOREM 5 Let $T \in L(X,Y)$. $T \in SF_+$ if and only if $T \in SRN_+$ and the restriction $T|_M$ is in SF_+ for every subspace M of X with the RNP.

THEOREM 6 A Banach space X has an infinite dimensional subspace with the RNP if and only there exist a Banach space Y and an operator $T \in SRN_+(X,Y)$ which is not upper semi-Fredholm.

COROLLARY 7 For every separable Banach space X without infinite dimensional subspaces with the RNP, each semiembedding of X is an isomorphism. If, in addition, each subspace of X contains an unconditional basic sequence, then X is hereditarily c_o.

Next we study the dual semigroup SRN_-. Recall that a Banach space X is Asplund if and only if its separable subspaces have separable dual; or equivalently, the dual X' has the RNP [19].

DEFINITION 8 $SRN_-(X,Y) := \{ T \in L(X,Y) / T' \in SRN_+ \}$

$RN^d(X,Y) := \{ K \in L(X,Y) / K' \in RN \}$

PROPOSITION 9. (1) SRN_- is a semigroup stable under perturbations of the operator ideal RN^d.

(2). If $T \in SRN_-(X,Y)$, then $Y/\overline{R(T)}$ is an Asplund space.

THEOREM 10 Let $T \in L(X,Y)$ with closed range. If $Y/R(T)$ is Asplund, then $T \in SRN_-$.

Finally we characterize lower semi-Fredholm operators and spaces without infinite dimensional Asplund quotients.

For a subspace N of Y, q_N is the quotient map onto Y/N.

THEOREM 11 Let X, Y be Banach spaces, and $T \in L(X,Y)$.

(a) $T \in SF_-$ if and only if $T \in SRN_-$ and $q_N T \in SF_-$ for every subspace N of Y such that Y/N is an Asplund space.

(b) Y has no infinite dimensional Asplund quotients if and only if $SRN_-(Z,Y) = SF_-(Z,Y)$ for every Banach space Z.

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