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VARIETIES OF NILPOTENT LIE ALGEBRAS OF DIMENSION LESS THAN 8

by

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Let N^n be the variety of n-dimensional complex nilpotent Lie algebra laws. Let us consider the "change of basis" action of $GL(n,\mathbb{C})$ on N^n and let us denote by $\theta(\mu)$ the orbit of the law μ in N^n under the above action. If $n\leq 6$, it is well known (see [5],[6],[7]) that N^n is irreducible and there exists a rigid filiform law μ_n^1 in N^n (that is, the orbit of μ_n^1 is open), and therefore N^n is the Zariski closure of $\theta(\mu_n^1)$. Nevertheless, N^7 is irreducible (see [3],[6],[7]) and it contains no rigid nilpotent Lie algebra (see [1],[4]). In this note the classification of complex nilpotent Lie algebras of dimension 7 obtained in [4] is used to prove that N^7 has precisely two irreducible components, both of dimension 40, which are respectively the Zariski closures of the orbits of a family $(\{\mu_\alpha^1\}_{\alpha\in\mathbb{C}})$ of filiform algebras, and a family $(\{\mu_\alpha^2\}_{\alpha\in\mathbb{C}})$ of characteristically nilpotent algebras. It is also shown that in N^8 there exist at least two irreducible components intersecting the open subset of all filiform Lie algebras, and one of which is the closure of the orbit of a rigid law in N^8; this is a counterexample of the conjecture: "there exists no rigid nilpotent Lie algebra in N^n for $n\geq 7$: given by M. Vergne in [7].

I. THE VARIETY N^7

Considerer the families $\{\mu_\alpha^1\}_{\alpha\in\mathbb{C}}$ and $\{\mu_\alpha^2\}_{\alpha\in\mathbb{C}}$ of Lie algebra laws in N^7 given by

- $\mu_\alpha^1(X_1,X_i)=X_{i-1}$, $i=3,4,5,6,7$; $\mu_\alpha^1(X_4,X_7)=\alpha X_2$; $\mu_\alpha^1(X_5,X_6)=X_2$; $\mu_\alpha^1(X_5,X_7)=(1+\alpha)X_3$; $\mu_\alpha^1(X_6,X_7)=(1+\alpha)X_4$, with $\alpha\in\mathbb{C}$.
- $\mu_\alpha^2(X_1,X_i)=X_{i-1}$, $i=4,5,6,7$; $\mu_\alpha^2(X_2,X_6)=X_3$; $\mu_\alpha^2(X_2,X_7)=X_3+X_4$; $\mu_\alpha^2(X_5,X_7)=\alpha X_3$; $\mu_\alpha^2(X_6,X_7)=\alpha X_4+X_2$, with $\alpha\in\mathbb{C}$.

where the indefined products are supposed to be zero.

Proposition. *If $\alpha\neq\alpha'$ (respectively $\alpha\neq\pm\alpha'$), then the laws μ_α^1 and $\mu_{\alpha'}^1$ (resp. μ_α^2 and $\mu_{\alpha'}^2$) are not isomorphic.*

Proposition. The families $\{\mu_\alpha^1\}_{\alpha\in\mathbb{C}}$ and $\{\mu_\alpha^2\}_{\alpha\in\mathbb{C}}$ are rigid in the following sense: Any perturbation μ of a standard element μ_α^1 (resp. μ_α^2) is isomorphic to a law in the family $\{\mu_\alpha^1\}$ (resp. $\{\mu_\alpha^2\}$). Moreover, there is no contraction between different standard members of these families.

We recall that μ is a perturbation of a standard law μ_1 of N^n if μ N^n and the structure constants relative to a standard basis in $\mathbb{C}^n$ are infinitely close. Also, the standard law μ_1 N^n can be contracted on the standard law $\mu_2 \in N^n$ if is a $f \in GL(n,\mathbb{C})$ such that $f^{-1}\cdot\mu_1\cdot(f\times f)$ is a perturbation of μ_2.

Proposition. All nilpotent standard laws in N^7 can be perturbed into a law in the families $\{\mu_\alpha^1\}_{\alpha\in\mathbb{C}}$ or $\{\mu_\alpha^2\}_{\alpha\in\mathbb{C}}$.

Theorem: The closures of the orbits of $\{\mu_\alpha^1\}_{\alpha\in\mathbb{C}}$ and $\{\mu_\alpha^2\}_{\alpha\in\mathbb{C}}$ are the only irreducible components in N^7. Moreover, both components have dimension 40.

II. THE VARIETY N^8

Let μ be the following law:

$\mu(X_1,X_i)=X_{i-1}$, $i=3,4,5,6,7,8$; $\mu(X_4,X_7)=X_2$; $\mu(X_4,X_8)=X_2+X_3$;

$\mu(X_5,X_6)=-X_2$; $\mu(X_5,X_7)=-\frac{2}{5}X_2$; $\mu(X_5,X_8)=X_4+\frac{3}{5}X_3$;

$\mu(X_6,X_7)=-\frac{2}{5}X_3$, $\mu(X_6,X_8)=X_5+\frac{1}{5}X_4$; $\mu(X_7,X_8)=X_6+\frac{1}{5}X_5$.

Proposition. The law μ is the only rigid filiform law in N^8 (if [2]). Therefore, the closure of $\theta(\mu)$ is an irreducible component of N^8.

Considerer the family of filiform algebras in N^8 defined by:

$\mu_\alpha(X_1,X_i)=X_{i-1}$, $i=3,4,5,6,7,8$; $\mu_\alpha(X_4,X_8)=\alpha X_2$; $\mu_\alpha(X_5,X_7=X_2$;

$\mu_\alpha(X_5,X_8)=(1+\alpha)X_3+X_2$; $\mu_\alpha(X_6,X_7)=X_3$; $\mu_\alpha(X_6,X_8)=(2+\alpha)X_4+X_3$;

$\mu_\alpha(X_7,X_8)=(2+\alpha)X_5+X_4$ with $\alpha\in\mathbb{C}\setminus\{-\frac{2}{5}\}$.

Proposition. The laws in the preceding family are not isomorphic. Moreover, no standard law in this family can be perturbed on μ. Therefore, the variety N^8 has at least two irreducible components which interseet the open subset of filiform algebras.

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