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SEMI-FREDHOLM OPERATORS AND SEMIGROUPS ASSOCIATED WITH SOME CLASSICAL OPERATOR IDEALS

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Several classical operator ideals are usually defined, or characterized, by means of the action of their operators on a class of bounded sequences, which are transformed in sequences with some convergence condition. This happens with the ideals Co, WCo, Ro, CC, and WCC of all compact, weakly compact, Rosenthal or l_1-singular, completely continuous and weakly completely continuous operators, respectively.

In fact, denoting X,Y Banach spaces, L(X,Y) the space of all continuous linear operators of X in Y and $T \in L(X,Y)$, recall that:

T belongs to Co (resp. WCo, Ro) if T sends bounded sequences into sequences which have a convergent (resp. weakly convergent, weak Cauchy) subsequence.

T belongs to CC (WCC) if T maps weak Cauchy sequences into convergent (weakly convergent) subsequences; note that $T \in$ CC if and only if T maps weakly convergent sequences into convergent sequences [7; 1.6.3].

For such an operator ideal U we shall associate a semigroup SU_+ of operators in the following way:

T belongs to SCo_+ (resp. $SWCo_+$, SRo_+) if every bounded sequence (x_n) in X such that (Tx_n) is convergent (resp. weakly convergent, weak Cauchy) has a convergent (resp. weakly convergent, weak Cauchy) subsequence.

T belongs to SCC_+ ($SWCC_+$) if every weak Cauchy sequence (x_n) in X such that (Tx_n) converges (weakly converges) is convergent (weakly convergent).

Clearly $SCo_+ \subset SU_+$ and SU_+ is semigroup in the sense that it is stable by products. In particular SCo_+ is the semigroup SF_+ of all upper semi-Fredholm operators [3; 4.1], $SWCo_+$ is the semigroup of all tauberian operators [3; 3.2], [11], and SCC_+ is the semigroup of all operators preserving mere Cauchy sequences considered in [5].

In this note we show interesting links between the ideal U and the associated semigroup SU_+. The main result says that every operator in SU_+ has kernel in the Banach space ideal Sp(U) of all Banach spaces E such that the identity operator I_E belongs to U; and the converse is true for operators with closed range.

Then we characterize SF_+ as the subset of operators in SU_+ such that the restrictions to subspaces in Sp(U) are in SF_+. Moreover, the incomparability class $Sp(U)^i$ of Sp(U) [1], formed by all Banach spaces totally incomparable (without common infinite dimensional subspaces) with every space in Sp(U), is characterized by the coincidence for X in $Sp(U)^i$ and every Banach space Y of $SU_+(X,Y)$ with $SF_+(X,Y)$.

We note that Sp(Co), Sp(WCo), Sp(Ro), Sp(CC) and Sp(WCC) are the well-known Banach space ideals F, R, Nl_1, Schur and WSC of all finite dimensional, reflexive, without copy of l_1, Schur and weakly sequentially complete spaces, respectively. And F^i, R^i, ${Nl_1}^i$, $Schur^i$ are respectively the ideals B of all Banach spaces, VIR of all very irreflexive spaces or spaces without infinite dimensional reflexive subspaces, Sl_1 of all somewhat l_1 spaces or spaces whose infinite dimensional subspaces contain a copy of l_1, and Nl_1. WSC^i has not been considered in the literature.

Our study covers the well-known case U = Co and that of Kalton and Wilansky for U = WCo [3]; moreover, a conjecture of Martin and Swart [5] about a characterization of Nl_1 is solved in the affirmative.

Finally we observe that for a Hilbert space H Wolf characterized in [10] upper semi-Fredholm operators as those operators T in L(H) satisfying the following condition: Every sequence (x_n) in H such that (x_n) weakly converges to 0 and (Tx_n) converges to zero, it is convergent to 0.

Considering Banach spaces X,Y and $T \in L(X,Y)$, by means of a similar argument to that of [7;1.6.3] it is shown that T belongs to the semigroup SCC_+ if and only if T satisfies the above condition. In this form the class SCC_+ is a natural extension to Banach spaces of the operators verifying Wolf condition.

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