The Sum problem for Hilbert spaces.

lesús M.F. Castillo.

Departamento de Matemáticas. Universidad de Extremadura. Avda de Elvas s/n. 06071 Badajoz. España.

We consider the following problem: there are locally convex spaces E for which the sum space  $\bigoplus$  E is a subspace of some product  $E^I$ , and others for which such an embedding is not possible. Examples of the first kind are  $E=C^\infty(R)$  or  $[1_\infty \mu(1_\infty 1_1)]$ . Examples of the second kind are the spaces carrying the weak topology.

**Problem.** Characterize those locally convex spaces E such that  $\bigoplus_{N} E$  is (isomorphic to) a subspace of some product  $E^{I}$ .

In this paper we give a complete solution of this problem for  $Hilbert\ spaces$ :

Theorem. Let H be a Hilbert space. Then the locally convex sum  $\oplus$  H is a subspace of the product H<sup>J</sup> if and only if H is infinite—dimensional, I is countable, and card  $J \geqslant 2^{\frac{N_o}{c}}$ .

Obviously H needs to be infinite-dimensional since  $\bigoplus$  K does not carry the weak topology and therefore cannot be a subspace of K<sup>I</sup>. It is also clear that card J  $\geqslant$  2<sup>cardI</sup> since this is the cardinal of a base of neighborhoods of 0 of the sum space.

The proof can be divided into some propositions:

Proposition 1.  $\bigoplus_{N} l_2$  is a subspace of  $l_2^J$  when card  $J \gg 2^{\frac{1}{N_0}}$ 

Proposition 2.  $\bigoplus_{I} 1_{2}$  is not a subspace of any product of copies of  $1_{2}$ 

Then we only need to consider the possibility of an embedding of  $\bigoplus_{i=1}^{n} l_2(\Lambda)$  into  $l_2(\Lambda)^j$  I,  $\Lambda$ , J uncountable. We have:

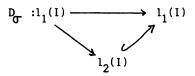
Lemma. Let I,J be uncountable sets, p > q and  $T:l_p(I) \longrightarrow l_q(J)$  a continuous operator. Then  $ImT \subset l_q(N)$ .

The preceding lemma asserts that such a T can only "move" a countable number of coordinates in J. From this follows:

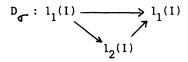
Lemma. A diagonal operator  $D_{\sigma}: l_p(I) \longrightarrow l_p(I)$ ,  $\sigma_i > 0 \quad \forall i \in I$  cannot be continuously factorized through  $l_q(I)$  if  $p \neq q$ .

Proposition 3. Let I be an uncountable set, and H a Hilbert space, then  $\bigoplus_{x}$  H is not a subspace of any product H<sup>J</sup>.

The idea is that if  $\bigoplus_I \mathbf{1}_2(I)$  is a subspace of  $\mathbf{1}_2(I)^J$  then a subfactorization of the diagonal operator



( $\Gamma_i > 0$  for all i) through  $l_2(I)$  could be done. Using the orthogonal projection onto  $\overline{ImA}$  we get a factorization



but then, by the lemma,  ${\rm ImD}_{\sigma} \subset 1_1(N),$  which is impossible since D\_ "moves" all the indexes.

## References

- |1| Jesús M.F.Castillo. The Sum problem for Banach spaces. Colloquium lectures 1988. Universidad de Extremadura.
- |2| H.Jarchow. Locally Convex Spaces. B.G.Teubner Stuttgart 1981.

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