SMOOTH TORAL ACTIONS ON PRINCIPAL BUNDLES AND
CHARACTERISTIC CLASSES

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The purpose of our work is to find explicit formulae for the computation of some
characteristic classes of smooth principal bundles $\mathcal{P}: P \to B$, in terms of local invariants
at a "singular subset" $A_G$ of $B$, associated to a smooth action of a compact Lie group
$G$ on $\mathcal{P}$. This singular subset, $A_G$, is defined as the set of points $x$ in $B$ whose
isotropy subgroups $G_x$ have dimension at least one.

The starting point for our research is the following result:

Let $\alpha \in H^2(\mathcal{P}(B;k)$ be a characteristic class of $\mathcal{P}$ with $2p > n - r$ ($n = \dim B,$
$r = \dim G$), where $H^*$ denotes singular cohomology and $k$ is any field of
characteristic zero. There exists then $\beta \in H^2(\mathcal{P}(B, B - A_G; k$) such that $J^*(\beta) = \alpha,$
where $J^*$ is the homomorphism induced in cohomology by the inclusion
$J: B \to (B, B - A_G)$. In particular, if the action of $G$ on $B$ is almost free, $\alpha$ must be
zero. (See [6] and [9] for the particular case of vector bundles)

If $B$ is compact and oriented, we can see more explicitly the dependence of the
characteristic classes on the singular set $A_G$, because in this case we have a
commutative diagram

$$
\begin{array}{ccc}
H^2(\mathcal{P}(B, B - A_G)) & \xrightarrow{J^*} & H^2(\mathcal{P}(B)) \\
\gamma \uparrow \cong & & \cong \uparrow \text{Poincaré duality} \\
H_n, 2p(A_G) & \longrightarrow & H_n, 2p(B)
\end{array}
$$

(see lemma 14, section 10, chapter 6 of [10], for the definition of $\gamma$), and then, if
$2p > n - r$, the Poincaré dual of $\alpha$ can be represented by a cycle $z$ of $A_G$. The general
problem is to find an explicit formula giving such a $z$. This kind of residue formula
should involve only the restriction of $\mathcal{P}$ to $A_G$, the action of $G$ on this restriction,
the embedding of $A_G$ in $B$ ("normal bundle" of $A_G$ in $B$), and the action of $G$ in this
"normal bundle".

We shall restrict ourselves to study the case of $G$ being a torus. This restriction
includes the equivalent formulation, for compact manifolds, of infinitesimal
isometries. We further assume that the action of $G$ on $B$ has finite orbit type (i.e. the
action has only a finite number of isotropy subgroups). This is the case, for instance,
when $B$ is compact (see [8]).

We have: 

$$A_G = \bigcup_{F \in \mathcal{F}} F$$

where $\mathcal{F}$ is the family of connected components of the fixed point sets under the action of all subtori $H$ of $G$, with $\dim H \geq 1$, appearing as 1-component of isotropy subgroups under the action of $G$ on $B$.

We need to assume a hypothesis concerning the "genericity" of the action:

**Definition 1.** The action of $G$ on $B$ is called generic if for each connected component of $A_G$, there exist $r$ subtori of $G$ of dimension one, $S_1, \ldots, S_r$, such that they generate $G$, i.e. $S_1 \cdots S_r = G$, and any subtorus of dimension one appearing as 1-component of isotropy subgroup on that connected component, is one of the $S_i$ (cf. 2.10 pag 42 of [1]).

In particular, the genericity assumption implies that, each element of $\mathcal{F}$ which is fixed by a subtorus of dimension $s$, is contained in exactly $s$ elements of $\mathcal{F}$ which are fixed only by subtori of dimension 1.

To construct characteristic classes, we use the Chern-Weil homomorphism of $\mathcal{F}$,

$$w_p : \text{Sym} \left( \mathcal{K} \right)_1 \to H^*_G(B)$$

(see for instance [7]), where $\text{Sym} \left( \mathcal{K} \right)_1$ is the graded algebra of multilinear symmetric functions in the Lie algebra $\mathcal{K}$ of the structure group $K$ of $\mathcal{F}$, invariant under the adjoint representation of $K$ in its Lie algebra $\mathcal{K}$.

**The residue classes.**

Let $F \in \mathcal{F}$ be a connected component of the fixed point set by a subtorus of dimension $s$, and let $F_1, \ldots, F_s$ be the elements of $\mathcal{F}$ containing $F$, and fixed only by subtori of dimension 1, $S_1, \ldots, S_r$, respectively.

Choose $h_i \neq 0$ in the Lie algebra $\mathcal{K}_i$ of $S_i$, $i = 1, \ldots, s$. Give $\eta_{F_i}$ (normal bundle of $F_i$ in $B$) the orientation induced by the complex structure associated to $h_i$, and give $\eta_F = \eta_{F_1} \otimes \cdots \otimes \eta_{F_s}$ the direct sum orientation. (See [6]).

Then, if $G \in \text{Sym} \left( \mathcal{K} \right)_1$, and $2m = \text{codim } F$, we consider the following "Laurent polynomial" in the indeterminates $X_1, \ldots, X_s$:

$$w \left( \left. \mathcal{P} \right|_F, \sum_{i=1}^s X_i h_i, \Gamma \right) \frac{(-2\pi)^m (-1)^n+1}{\prod_{i=1}^n w \left( \eta_{F_i} \left| F \right|, X_i h_i, Pf_{F_i} \right)}$$

where $w \left( \xi, h, \right)$ denotes the "generalized" Chern-Weil homomorphism (see [2], or
where the definition differs from the one in this paper by a certain constant factor.

**Definition 2.** We define the residue class \( \alpha_\Gamma(F) \in H_{\text{rk}}^{2p-2m}(F) \) as the coefficient of the term of degree 0 of the above "Laurent polynomial".

Now, we can state the residue formula as follows:

**Theorem.** Let \( \mathcal{P}: P \to B \) be a smooth principal bundle with structural group \( K \), and assume that a torus \( G \) acts smoothly on \( \mathcal{P} \). Suppose that the action of \( G \) on \( B \) has a finite number of orbit types and it is generic. Then, if \( 2p > \dim B - \dim G \),

\[
w_p(\Gamma) = \sum_{F \in \mathcal{F}} \int_{-1}^{-1} \alpha_\Gamma(F),
\]

for \( \Gamma \in \text{Sym}^p(K) \), and where \( \int_{-1}^{-1} \) denotes the inverse of the fiber integral \( \int \) associate to any tubular neighborhood \( U \) of \( F \) in \( B \) oriented as above, followed by the canonical homomorphism \( H_{\text{rk}}^\bullet(U) \to H^\bullet(B) \).

For the proof, see [2].

Residue formulae for characteristic classes in some particular cases have been found by other authors. See for instance [3], [6],[9] in the real case, [4] in the complex case, and [5] for \( \mathbb{Z}/(2) \) coefficients.

**REFERENCES**


