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MALCEV H*-ALGEBRAS

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Associative H*-algebras were studied by Ambrose in [1] and alternative H*-algebras were studied by Pérez de Guzmán in [4]. The algebra obtained by symmetrization of the product of an alternative H*-algebra is a Jordan H*-algebra. These algebras were studied by Viola [9], Viola and Rema [10] and Cuenca and Rodríguez in [2]. The algebra obtained by skew-symmetrization of an alternative H*-algebra is a Malcev H*-algebra, in particular one obtains, if the initial algebra is associative, Lie H*-algebras which were studied by Schue [7,8]. In this paper we deal with Malcev H*-algebras.

A Malcev algebra is defined to be a nonassociative algebra A satisfying the identities

$$xy=-yx \quad \text{and} \quad J(x,y,xz)= J(x,y,z)x$$

for all x,y,z in A, where

$$J(x,y,z)=(xy)z +(yz)x +(zx)y$$

Sagle [5] studies in full detail the finite-dimensional Malcev algebras and for any algebraically closed field $\mathbb{K}$ of caracteristic zero he introduces a Malcev algebra over $\mathbb{K}$ A_7 which is known to be the only finite dimensional simple non Lie Malcev algebra over $\mathbb{K}$ [3,6].

A H*-algebra is a complex nonassociative algebra A whose underlying vector space is a Hilbert space, together with an algebra involution * such that

$$(xy/z)=(x/zy^*)=(y/x^*z) \text{ for all } x,y,z \text{ in } A$$

A H*-algebra A is called topologically simple if $A^2\neq 0$ and A contains no nonzero proper closed ideals.

In [2] Cuenca and Rodríguez started the structure theory for non-associative H*-algebras, showing that every non-associative H*-algebra with zero annihilator is the closure of the orthogonal sum of its minimal closed ideals, each of which is a topologically simple H*-algebra. This result reduces the task of description of the H*-algebras of any hereditary class to the topologically simple case.

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The seven-dimensional simple non-Lie Malcev complex algebra A_7 introduced by Sagle as the quotient of $\mathbb{O}^-$ (the akew-symmetrization of the algebra $\mathbb{O}$ of complex octonions) over its annihilator ideal $\mathbb{C}1$, can be structured in a natural way as a H*-algebra, since $\mathbb{O}$ can be endowed with a H*-algebra structure [4].

Our main result is the following

THEOREM. Let A be a topologically simple non-Lie Malcev H*-algebra. Them A is, up to a positive scalar multiple of the inner product, the simple non-Lie Malcev H*-algebra A_7.

The strategy of our proof is to show that the quadratic H*-algebra D, associated to the algebra A in our theorem, is an alternative algebra, where $D=C \oplus A$ with product, involution $\square$ and inner product defined by

$$(\alpha,x)(\beta,y)=(\alpha\beta+(x/y^*),\beta x+\alpha y+xy),\ (\alpha,x)^{\square}=(\bar{\alpha},x^*)$$

$$((\alpha,x)/(\beta,y))=\alpha\bar{\beta}+(x/y)$$

for α,β complex numbers and x,y elements in A. For this purpose, we show that the algebra A satisfies (up to a positive scalar multiple of the inner product, if necessary) the equality

(1) $L_x^2 = (x/x^*)I-x \otimes x^*$ for all x in A, where L_x denotes the left multiplication operator by x in A, I denotes the identity operator on A and, as usual, $x \otimes x^*(y)=(y/x^*)x$ for all y in A.

It is easily verified that D is alternative if and only if the above equality holds.

The proof of the equality (1) is obtained by suitable infinite dimensional extensions of the arguments of Sagle in [6] for which the Cartan decomposition in Lie H*-algebras given by schue [7] becomes crucial.

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