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BIDUALS OF p - LATTICE SUMMING OPERATORS

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Let E be a Banach space and F be a Banach lattice. A (linear, continuous) operator $T: E \to F$ is said to be a p - lattice summing operator ($1 \leqslant p \leqslant \infty$) if there exists a constant $K \geqslant 0$ such that for every finite family $\{x_1,\ldots,x_n\}$ in E we have:

(1) $||(\ \Sigma_{i=1}^{n}\ |Tx_i|^p)^{1/p}|| \leqslant\ K \sup\ \{(\ \Sigma_{i=1}^{n}\ |\langle x_i,x'\rangle|\ ^p)^{1/p}, x'\epsilon B_{E'}\}$

where $B_{E'}$ is the unit ball in E', and $(\ \Sigma_{i=1}^{n}\ |Tx_i|^p)^{1/p}$, given by the Krivine calculus for 1 - homogeneous continuous expresions ({1}), can be writen in the form:

(2) $(\ \Sigma_{i=1}^{n}\ |Tx_i|^p)^{1/p} = \sup\ \{\ \Sigma_{i=1}^{n}\ a_i Tx_i,\ a_1,\ldots,a_n \epsilon R,\ \Sigma_{i=1}^{n}|a_i|^q \leqslant 1\}$

(where $1/p + 1/q = 1$). The smallest constant K which verifies (1) is denoted by $\lambda_p(T)$.

This class of operators is a natural extension of the p - summing operators defined by Fiestch (see {2}).

In this paper we present the following result:

"*If E is a Banach space and F is a Banach lattice, then T: E → F is a p - lattice summing operator if and only if T": E" → F" is also a p - lattice summing operator*".

The proof is based on the Local Reflexivity Principle ({3}), and on some results of lattice theory.

Local Reflexivity Principle:

Let G be a finite subspace of the bidual E" of a Banach space E, and H be a finite subspace of the dual E'. Given $t \geqslant 0$, there exists an operator $R: G \to E$ such that: i) $||R|| \leqslant 1 + t$; ii) $\langle Rx'',y'\rangle = \langle x'',y'\rangle$ for every $x'' \in G$ and $y' \in H$; and iii), if $J: E \to E''$ is the canonical inclusion, then for every x" in $G \cap J(E)$, $J \bullet Rx'' = x''$.

Proof of the result:

Let E be a Banach space and F be a Banach lattice. Obviously, if $T'': E'' \to F''$ is p - lattice summing, then $T: E \to F$ must be p - lattice summing also. So, consider $x''_1,\ldots,x''_n$ in E"; we need to find a suitable estimation for $||(\Sigma_{i=1}^{n} |T''x''_i|^p)^{1/p}||$.

For each finite set C in the unit ball Bq of $(\mathbb{R}^q, ||.||_q)$, define $y''_C = \sup \{ \Sigma_{i=1}^{n} a_i T''x''_i,\ a=(a_1,\dots,a_n) \in C\cup\{0\} \}$. Then, by (2), we have $(\Sigma_{i=1}^{n} |T''x''_i|^p)^{1/p} = \sup_C y''_C$. The family $\{y''_C\}$ is an increasing net of positive vectors in F'', norm bounded (by the norm of the supremum), and therefore it is $\sigma(F'',F''')$ - Cauchy and $\sigma(F'',F')$ - relatively compact. In particular $\{y''_C\}$ has an accumulation point y''_o for the $\sigma(F'',F')$ topology, and y''_C converges to y''_o in $\sigma(F'',F')$. It can be shown that $y''_o = \sup_C y''_C$.

Hence, given $\varepsilon \geq 0$ there exists $y' \in F'$ such that $y' \geq 0$, $||y'|| \leq 1$ and

$$||(\Sigma_{i=1}^{n} |T''x''_i|^p)^{1/p}|| = ||y''_o|| \leq \langle y''_o, y'\rangle + \varepsilon = \lim_C \langle y''_C, y'\rangle + \varepsilon$$
$$\leq \langle y''_{C_o}, y'\rangle + 2\varepsilon$$

for some finite set $C_o = \{a^1,\dots,a^m\}$ in Bq, with $0 \in C_o$. By {4}, II,5,5 and II,4,2:

$$\langle y''_{C_o}, y'\rangle = \langle \sup \{\Sigma_{i=1}^{n} a_i^j T''x''_i,\ 1 \leq j \leq m\}, y'\rangle =$$
$$= \sup \{\Sigma_{j=1}^{m} (\Sigma_{i=1}^{n} a_i^j \langle T''x''_i, y'_j\rangle),\ y'_1 + \dots + y'_m = y',\ y'_j \geq 0\} \leq$$
$$\leq \Sigma_{j=1}^{m} (\Sigma_{i=1}^{n} a_i^j \langle x''_i, T'y'_j\rangle) + \varepsilon$$

for some $y'_1,\dots,y'_m$ in F' with $y'_j \geq o$, $1 \leq j \leq m$, and $y'_1 + \dots + y'_m = y'$.

We use now the local reflexivity principle, with $G = [x''_1,\dots,x''_n] \subset E''$, and $H = [T'y'_1,\dots,T'y'_m] \subset E'$: given $t \geq 0$, there is an operator $R: G \to E$ such that $||R|| \leq 1 + t$, and $\langle x''_i, x'\rangle = \langle Rx''_i, x'\rangle$ for every $x' \in H$. Writing $x_i = Rx''_i$: $\langle x''_i, T'y'_j\rangle = \langle x_i, T'y'_j\rangle = \langle Tx_i, y'_j\rangle$, $1 \leq i \leq n$, $1 \leq j \leq m$. Then:

$$||y''_o|| \leq \Sigma_{j=1}^{m}(\Sigma_{i=1}^{n} a_i^j \langle Tx_i, y'_j\rangle) + 3\varepsilon \leq$$
$$\leq \langle \sup \{ \Sigma_{i=1}^{n} a_i^j Tx_i,\ 1 \leq j \leq m\},\ \Sigma_{j=1}^{m} y'_j \rangle + 3\varepsilon \leq$$
$$\leq ||(\Sigma_{i=1}^{n} |Tx_i|^p)^{1/p}|| + 3\varepsilon.$$

As, by hypothesis, T is p - lattice summing,

$$||y''_o|| \leq \lambda_p(T) \sup \{(\Sigma_{i=1}^{n} |\langle x_i, x'\rangle|^p)^{1/p},\ x' \in B_{E'}\} + 3\varepsilon.$$

Finally, for each x' in $B_{E'}$, $|\langle x_i, x'\rangle| = |\langle Rx''_i, x'\rangle| = |\langle x''_i, R'x'\rangle|$ where $R'x' : G \to K$ is a continuous linear form with $||R'x'|| \leq 1 + t$. By the Hahn - Banach Theorem, there exists an extension x''' of x' to E'', with $||x'''|| \leq 1 + t$: and therefore

$$|\langle x_i'',R'x'\rangle| = |\langle x_i'',x'''\rangle| = (1+t)\ |\langle x_i'',\frac{x'''}{1+t}\rangle|$$

where $x'''/(1+t) \in B_{E'''}$.

Consequently

$$\sup\{(\Sigma_{i=1}^{n} |\langle x_i,x'\rangle|^p)^{1/p},\ x' \in B_{E'}\} \leqslant$$
$$\leqslant (1+t)\sup\{(\Sigma_{i=1}|\langle x_i'',x'''\rangle|^p)^{1/p},\ x''' \in B_{E'''}\}$$

We have obtained that for every $\varepsilon \geqslant 0$ and every $t \geqslant 0$

$$||(\Sigma_{i=1}^{n} |T''x_i''|^p)^{1/p}|| \leqslant \lambda_p(T)\ (1+t)\sup\{(\Sigma_{i=1}^{n}|\langle x_i'',x'''\rangle|^p)^{1/p},$$
$$x''' \in B_{E'''}\} + 3\,\varepsilon$$

and then T'' is p - lattice summing, with $\lambda_p(T'') \leqslant \lambda_p(T)$.

Remarks:

1- The result is true also for $p = \infty$ ($T: E \to F$ is said to be ∞ - lattice summing if there is a constant $K \geqslant 0$ such that for every finite family $\{x_1,\dots, x_n\}$ in E

$$||\vee_{i=1}^{n} |Tx_i|\ || \leqslant K \max\{||x_i||,\ 1 \leqslant i \leqslant n\}),$$

and the proof is very similar.

2- The same technics can be used to prove the well known result of Piestch which states that if T is a (p, q) - summing operator from a Banach space E into a Banach space F (i.e. $\exists K \geqslant 0$ such that $\forall\ x_1,\dots x_n$ in E, $(\Sigma_{i=1}^{n} ||Tx_i||^q)^{1/q} \leqslant K \sup\{(\Sigma_{i=1}^{n} |\langle x_i,x'\rangle|^p)^{1/p},\ x' \in B_{E'}\}$), then the bidual $T'': E'' \to F''$ is also (p, q) - summing: for $x_1'',\dots, x_n''$ in E'', one has just to bound $(\Sigma_{i=1}^{n} ||T''x_i''||^q)^{1/q}$ with other expresion of the form $(\Sigma_{i=1}^{n} |\langle T''x_i'',y_i'\rangle|^q)^{1/q} +\varepsilon$, with $y_i' \in F'$, $||y_i'|| \leqslant 1, 1 \leqslant i \leqslant n$. ({3}, {5}).

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